Ageing, dynamical scaling and its extensions in many-particle systems without detailed balance

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Abstract. Recent studies on the phenomenology of ageing in certain many-particle systems which are at a critical point of their non-equilibrium steady-states, are reviewed. Examples include the contact process, the parity-conserving branching-annihilating random walk, two exactly solvable particle-reaction models and kinetic growth models. While the generic scaling descriptions known from magnetic system can be taken over, some of the scaling relations between the ageing exponents are no longer valid. In particular, there is no obvious generalization of the universal limit fluctuation-dissipation ratio. The form of the scaling function of the two-time response function is compared with the prediction of the theory of local scale-invariance.

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1. Introduction

Symmetry principles have been extremely useful for the understanding of complex many-body systems, where the interactions between the degrees of freedom are sufficiently strong as to render perturbative methods inapplicables. Here we are interested in the slow non-equilibrium dynamics shown by many-body systems which are rapidly brought out from some initial state ('quenched') to a region in phase-space where either the equilibrium state naturally generates a slow dynamics (this is for example realized for systems *at* a critical point) or else into a coexistence region dominated by several equivalent stationary states. One of the essential features of such systems is that their properties depend on their 'age', that is the time elapsed since the quench. Of course, any biological system ages, but there is also 'physical ageing' which arises even if the underlying microscopic dynamics is completely reversible. One might formally define *ageing* by this breaking of time-translation invariance, associated with a slow dynamics which generically leads to some form of dynamical scaling. Physical ageing was originally seen to occur in glassy systems [120] and has been used since prehistoric times by engineers in the processing of materials. Quite recently, it has been realized that very similar phenomena can also be found in simple magnets, without disorder nor frustrations. The study of these supposedly simpler systems may lead to conceptual insights which in turn could become also fruitful in more complex systems. The topic has been under intensive study, see [19, 33, 52, 32, 65, 84, 24, 68, 47, 70] for reviews. The reversibility of the microdynamics in systems undergoing physical ageing means that their stationary states are equilibrium states. In numerical simulations of such systems this is realized by choosing the dynamics such as to satisfy detailed balance. In many systems undergoing physical ageing, detailed balance and consequently the relaxation towards *equilibrium* steady-states is taken for granted, as there are many textbook proofs [123, 129] of detailed balance for closed, isolated systems.

On the other hand, it has become increasingly clear from studies in anomalous chemical kinetics that several of the constitutive properties of ageing are naturally met in many situations. First, it is well-known that fluctuation effects may lead to slow, non-exponential relaxation in *irreversible* chemical reactions - not accounted for by mean-field schemes, see e.g. [117, 64] for reviews and references therein. Second, it was understood more recently through the work of Oshanin and collaborators [97, 98, 6, 31, 124, 125] that even for *reversible* reactions a slow, non-exponential relaxation may generically occur without the fine-tuning of parameters and furthermore, that the steady-states to which relaxation occurs depend on the kinetic coefficients and hence cannot be equilibrium states. Consequently, detailed balance cannot be valid in these systems. While in these studies long-range interactions (as they may naturally arise in reactions of large molecules or in studies of radiation damage [6]) play an essential rôle, detailed balance need not be satisfied even in kinetic systems with contact interactions. For example, in the system defined by the simultaneous reversible reactions $2A \longleftrightarrow \emptyset$, $2A \longleftrightarrow A$,

$A \longleftrightarrow \emptyset$ with diffusive motion of single particles, detailed balance only holds is certain conditions on the reaction rates are met [4]. Since detailed balance is already found to be broken in very simple reactions such as $2A \longleftrightarrow B$ [124, 125] or $A+B \longrightarrow \emptyset$ [6, 31], it is conceivable that the phenomenon might be much more common. Furthermore, since sometime a slow relaxation in kinetic models is brought into relationship with glassy dynamics [90], it is of interest to investigate to what extent the three essential properties of physical ageing – slow dynamics, dynamical scaling and breaking of time-translation invariance – are actually realized in chemical kinetics.

Therefore, we shall review here recent progress about ageing in many-body systems with a more general dynamics where detailed balance is no longer required to hold and therefore non-equilibrium steady-states may arise. Since there are as yet only few studies available on these systems, comparison with ageing in simple magnets should be a useful guide. For a similar reason, we shall investigate the behaviour of models whose physical origin is very different which should lead to some insight about generic properties. Because of the possibility of non-equilibrium steady-states, the systems under consideration here are closer to biological/chemical ageing than those considered up to now in studies of physical ageing. Remarkably, there is evidence that some dynamical symmetries recently discovered in physical ageing may also extend to this more general class of systems.

We shall define the systems we want to study in the next section. Before we come to that, however, we shall briefly recall for reference some of the main results about the ageing of simple magnets. We assume throughout that the order-parameter is *non-conserved* by the dynamics and that the initial state is totally disordered, unless explicitly stated otherwise. Besides the breaking of time-translation invariance, ageing systems are often characterized by dynamical scaling. It has become common to study ageing behaviour through the two-time autocorrelation and (linear) autoresponse functions

$$C(t,s) = \langle \phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r})\rangle \quad \sim s^{-b} f_C(t/s) \tag{1}$$

$$R(t,s) = \left.\frac{\delta\langle\phi(t,\boldsymbol{r})\rangle}{\delta h(s,\boldsymbol{r})}\right|_{h=0} \sim s^{-1-a} f_R(t/s) \tag{2}$$

where $\phi(t,\boldsymbol{r})$ is the order-parameter at time t and location $\boldsymbol{r}$ and $h(s,\boldsymbol{r})$ is the conjugate magnetic field at time s and location $\boldsymbol{r}$. The scaling behaviour is expected to apply in the so-called *ageing regime* where $t, s \gg t_{\rm micro}$ and $t-s \gg t_{\rm micro}$, where $t_{\rm micro}$ is a microscopic time-scale. We illustrate this in figure 1 which shows the autocorrelation function $C(t,s)$ of the $3D$ Ising model with (non-conserved) Glauber dynamics† [49] quenched from a fully disordered initial state to a temperature $T < T_c$. When plotting the data over against $t-s$, we see that the data depend on *both* $t-s$ and s, hence time-translation invariance is broken and the system ages. Further, with increasing values of the waiting time s, the system becomes 'stiffer' and a plateau close to the

† Recall that the zero-temperature Glauber model can be mapped, via a duality transformation [118] or a similarity transformation [111], to the kinetic model $2A \longrightarrow \emptyset$ with single-particle diffusion.

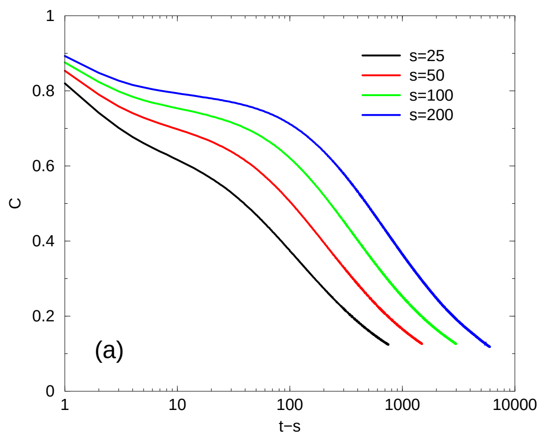

Figure 1. (a) Ageing and (b) dynamical scaling of the two-time autocorrelation function $C(t,s)$ in the $3D$ Glauber-Ising model quenched to $T = 3 < T_c$, for several values of the waiting time s [105].

equilibrium value $C_{\rm eq} = M_{\rm eq}^2$ develops when $t-s$ is not too large before the correlations fall off rapidly when $t-s \to \infty$. When replotting the same data over against t/s, a data collapse is found if s is large enough which is evidence for dynamical scaling.

The distance of such systems from a global equilibrium state can be measured through the fluctuation-dissipation ratio, defined as [34]

$$X(t,s) := TR(t,s) \left(\frac{\partial C(t,s)}{\partial s} \right)^{-1} \tag{3}$$

At equilibrium, $X(t,s) = 1$ from the fluctuation-dissipation theorem. One often considers the limit fluctuation-dissipation ratio $X_\infty := \lim_{s\to\infty} \left(\lim_{t\to\infty} X(t,s) \right)$.‡ For quenches to below T_c, one usually has $X_\infty = 0$ but for *critical* quenches onto $T = T_c$, it has been argued by Godrèche and Luck that X_∞ should be a *universal* number [51], since it can be written as a ratio of two scaling amplitudes. This universality has been thoroughly confirmed for systems relaxing towards equilibrium steady-states, see [32, 24] for recent reviews.

Furthermore, in writing eqs. (1,2) it was tacitly assumed that the scaling derives from the algebraic time-dependence of a single characteristic length-scale $L(t) \sim t^{1/z}$ which measures the linear size of correlated or ordered clusters and where z is the dynamic exponent. For a $2D$ Glauber-Ising model quenched to $T = T_c$ the growth of correlated clusters in illustrated in figure 2 where the black/white site represent the two states of the Ising spins. Then the above forms define the non-equilibrium exponents a and b and the scaling functions $f_C(y)$ and $f_R(y)$. For large arguments $y \to \infty$, one generically expects

‡ The order of the limits is crucial, since $\lim_{t\to\infty} \left(\lim_{s\to\infty} X(t,s) \right) = 1$.

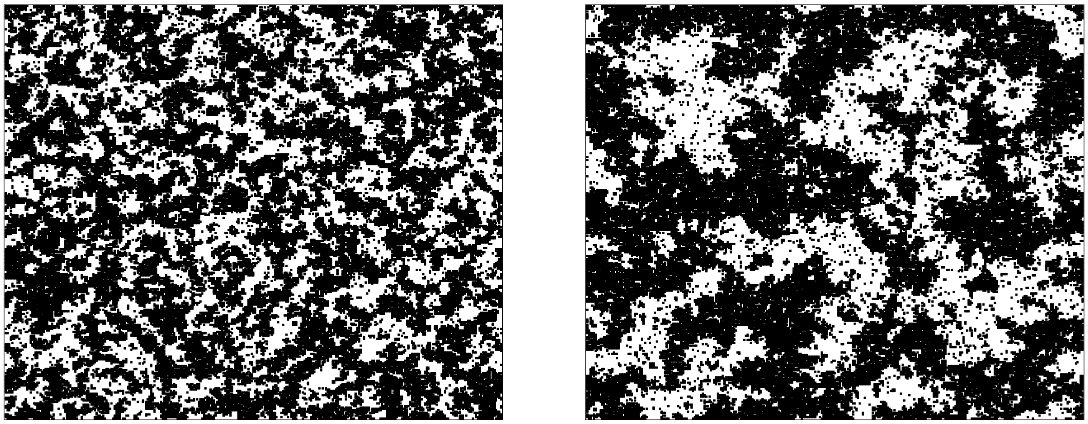

Figure 2. Snapshots from the $2D$ Glauber-Ising model quenched to $T = T_c$ from a disordered initial state at (left panel) $t = 25$ and (right panel) $t = 275$ MC time steps after the quench [105].

$$f_C(y) \sim y^{-\lambda_C/z} \quad , \quad f_R(y) \sim y^{-\lambda_R/z} \tag{4}$$

where λ_C and λ_R, respectively, are known as autocorrelation [46, 79] and autoresponse exponents [103]. While in non-disordered magnets with short-ranged initial conditions one usually has $\lambda_C = \lambda_R$, this is not necessarily so if either of these conditions is relaxed. From a field-theoretical point of view it is known that for a non-conserved order-parameter the calculation of $\lambda_{C,R}$ requires an independent renormalization and hence one cannot expect to find a scaling relation between these and equilibrium exponents (including z) [81]. On the other hand, the values of the exponents a and b are known. For quenches to $T = T_c$, the relevant length-scale is set by the time-dependent correlation length $L(t) \sim \xi(t) \sim t^{1/z}$ and this leads to $a = b = (d-2+\eta)/z$, where η is a standard equilibrium exponent. For quenches into the ordered phase $T < T_c$, one usually observes simple scaling of $C(t,s) = f_C(t/s)$, hence $b = 0$.§ The value of a depends on whether the equilibrium correlator is short- or long-ranged, respectively. These may be referred to as classes S and L, respectively and one has, see e.g. [30, 59, 63]

$$C_{\rm eq}(\boldsymbol{r}) \sim \left\{ \begin{array}{l} e^{-|\boldsymbol{r}|/\xi} \\ |\boldsymbol{r}|^{-(d-2+\eta)} \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} \text{class S} \\ \text{class L} \end{array} \right. \Longrightarrow a = \left\{ \begin{array}{c} 1/z \\ (d-2+\eta)/z \end{array} \right. \tag{5}$$

Examples for short-ranged models (class S) include the Ising or Potts models in $d > 1$ dimensions (and $T < T_c$), while all systems quenched to criticality, or the spherical model or the $2D$ XY model below the Kosterlitz-Thouless transition are examples for long-ranged systems (class L).

In equilibrium critical phenomena, it is well-known that the standard scale-invariance can, under quite weak conditions, be extended to a *conformal* invariance. Roughly, a conformal transformation is a scale-transformation $\boldsymbol{r} \mapsto b\boldsymbol{r}$ with a space-dependent rescaling factor $b = b(\boldsymbol{r})$ (such that angles are kept unchanged). In particular,

§ This needs no longer be the case when the ageing close to a free surface is considered [10].

in two dimensions conformal invariance allows to derive from the representation theory of the conformal (Virasoro) algebra the possible values of the critical exponents, to set up a list of possible universality classes, calculate explicitly all n-point correlation functions and so on [16, 28]. One might wonder whether a similar extension might be possible at least in some instances of dynamical scaling and further ask *whether response functions or correlation functions might be found from their covariance under some generalized dynamical scaling with a space-time-dependent rescaling factor* $b = b(t, \boldsymbol{r})$ [56, 58] ? We shall discuss the question here in the specific context of ageing and shall focus on what can be said about the scaling functions $f_{C,R}(y)$ in a model-independent way.

A useful starting point is to consider the symmetries of the free diffusion (or free Schrödinger) equation

$$2\mathcal{M}\partial_t\phi = \Delta\phi \tag{6}$$

where $\Delta = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$ is the spatial laplacian and the 'mass' $\mathcal{M}$ can be seen as a kinetic coefficient. Indeed, it was already shown by Lie more than a century ago that this equation has more symmetries than the trivial translation- and rotation-invariances. Consider the so-called *Schrödinger-group* defined through the space-time transformations

$$t \mapsto t' = \frac{\alpha t + \beta}{\gamma t + \delta} \;\; ; \;\; \boldsymbol{r} \mapsto \boldsymbol{r}' = \frac{R\,\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{a}}{\gamma t + \delta} \;\; , \;\; \alpha\delta - \beta\gamma = 1 \tag{7}$$

where $\alpha, \beta, \gamma, \delta, \boldsymbol{v}, \boldsymbol{a}$ are real (vector) parameters and R is a rotation matrix in d spatial dimensions. The group acts projectively on a solution ϕ of the diffusion equation through $(t, \boldsymbol{r}) \mapsto g(t, \boldsymbol{r})$, $\phi \mapsto T_g\phi$

$$\left(T_g\phi\right)(t, \boldsymbol{r}) = f_g(g^{-1}(t, \boldsymbol{r}))\,\phi(g^{-1}(t, \boldsymbol{r})) \tag{8}$$

where g is an element of the Schrödinger group and the companion function reads [95, 102]

$$f_g(t, \boldsymbol{r}) = (\gamma t + \delta)^{-d/2} \exp\left[-\frac{\mathcal{M}}{2}\frac{\gamma \boldsymbol{r}^2 + 2\,R\,\boldsymbol{r}\cdot(\gamma\boldsymbol{a} - \delta\boldsymbol{v}) + \gamma\boldsymbol{a}^2 - t\delta\boldsymbol{v}^2 + 2\gamma\boldsymbol{a}\cdot\boldsymbol{v}}{\gamma t + \delta}\right] \tag{9}$$

It is then natural to include also arbitrary phase-shifts of the wave function ϕ within the Schrödinger group $Sch(d)$. In what follows, we denote by $\mathfrak{sch}_d$ the Lie algebra of $Sch(d)$. The Schrödinger group so defined is the largest group which maps *any* solution of the free Schrödinger equation (with $\mathcal{M}$ fixed) onto another solution. This is easily seen in $d = 1$ by introducing the Schrödinger operator

$$\mathcal{S} := 2M_0 Y_{-1} - Y_{-1/2}^2 \tag{10}$$

The Schrödinger Lie algebra $\mathfrak{sch}_1 = \langle X_{-1,0,1}, Y_{-\frac{1}{2},\frac{1}{2}}, M_0\rangle$ is spanned by the infinitesimal generators of temporal and spatial translations $(X_{-1}, Y_{-1/2})$, Galilei-transformations

($Y_{1/2}$), phase shifts (M_0), space-time dilatations with $z=2$ (X_0) and so-called special transformations (X_1). Explicitly, the generators read [56]

$$
\begin{aligned}
X_n &= -t^{n+1}\partial_t - \frac{n+1}{2}t^n r\partial_r - \frac{n(n+1)}{4}\mathcal{M}t^{n-1}r^2 - \frac{x}{2}(n+1)t^n \\
Y_m &= -t^{m+1/2}\partial_r - \left(m+\frac{1}{2}\right)\mathcal{M}t^{m-1/2}r \\
M_n &= -\mathcal{M}t^n
\end{aligned}
\tag{11}
$$

Here x is the scaling dimension and $\mathcal{M}$ is the mass of the scaling operator ϕ on which these generators act. The non-vanishing commutation relations are

$$
\begin{aligned}
[X_n, X_{n'}] &= (n-n')X_{n+n'} \quad , \quad [X_n, Y_m] = \left(\frac{n}{2}-m\right)Y_{n+m} \\
[X_n, M_{n'}] &= -n'M_{n+n'} \quad , \quad [Y_m, Y_{m'}] = (m-m')M_{m+m'}
\end{aligned}
\tag{12}
$$

The invariance of the diffusion equation under the action of $\mathfrak{sch}_1$ is now seen from the following commutators which follow from the explicit form (11)

$$
\begin{aligned}
[\mathcal{S}, X_{-1}] &= \left[\mathcal{S}, Y_{\pm 1/2}\right] = [\mathcal{S}, M_0] = 0 \\
[\mathcal{S}, X_0] &= -\mathcal{S} \quad , \qquad [\mathcal{S}, X_1] = -2t\mathcal{S} - (2x-1)M_0
\end{aligned}
\tag{13}
$$

Therefore, *for any solution ϕ of the Schrödinger equation $\mathcal{S}\phi = 0$ with scaling dimension $x = 1/2$, the infinitesimally transformed solution $\mathcal{X}\phi$ with $\mathcal{X} \in \mathfrak{sch}_1$ also satisfies the Schrödinger equation $\mathcal{S}\mathcal{X}\phi = 0$* [83, 95, 55]. For applications to ageing, we must consider to so-called *ageing algebra* $\mathfrak{age}_1 = \langle X_{0,1}, Y_{-\frac{1}{2},\frac{1}{2}}, M_0\rangle \subset \mathfrak{sch}_1$ (without time-translations) which is a true subalgebra of $\mathfrak{sch}_1$. Extensions to $d > 1$ are straightforward.

What is the usefulness of knowing dynamical symmetries of free, simple diffusion for the understanding of non-equilibrium kinetics ? One way of setting up the problem would be to write down a stochastic Langevin equation for the order-parameter. The simplest case is usually considered to be a dynamics without macroscopic conservation laws (model A), where one would have [76]

$$
2\mathcal{M}\frac{\partial\phi}{\partial t} = \Delta\phi - \frac{\delta\mathcal{V}[\phi]}{\delta\phi} + \eta
\tag{14}
$$

where $\mathcal{V}$ is the Ginzburg-Landau potential and η is a gaussian noise which describes the coupling to an external heat-bath and the initial distribution of ϕ. At first sight, there appear to be no non-trivial symmetries, because (14) cannot be Galilei-invariant, because of the noise term η. To understand this physically, consider a magnet which is a rest with respect to a homogeneous heat-bath at temperature T. If the magnet is moved with a constant velocity with respect to the heat-bath, the effective temperature will now appear to be direction-dependent, and the heat-bath is no longer homogeneous. However, this difficulty can be avoided as follows [104]: *split the Langevin equation into a 'deterministic' part with non-trivial symmetries and a 'noise' part and then show using these symmetries that all averages can be reduced exactly to averages within the deterministic, noiseless theory.* Technically, one first constructs in the standard fashion

(Janssen-de Dominicis procedure) [35, 82] the associated stochastic field-theory with action $J[\phi,\widetilde{\phi}]$ where $\widetilde{\phi}$ is the response field associated to the order-parameter ϕ. Second, decompose the action into two parts

$$J[\phi,\widetilde{\phi}] = J_0[\phi,\widetilde{\phi}] + J_b[\widetilde{\phi}] \tag{15}$$

where

$$J_0[\phi,\widetilde{\phi}] = \int_{\mathbb{R}_+\times\mathbb{R}^d} \mathrm{d}t\mathrm{d}\boldsymbol{r}\; \widetilde{\phi}\left(2\mathcal{M}\partial_t\phi - \Delta\phi + \frac{\delta\mathcal{V}}{\delta\phi}\right) \tag{16}$$

contains the terms coming from the 'deterministic' part of the Langevin equation ($\mathcal{V}$ is the self-interacting 'potential') whereas

$$J_b[\widetilde{\phi}] = -T\int_{\mathbb{R}_+\times\mathbb{R}^d} \mathrm{d}t\mathrm{d}\boldsymbol{r}\; \widetilde{\phi}(t,\boldsymbol{r})^2 - \frac{1}{2}\int_{\mathbb{R}^{2d}} \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\; \widetilde{\phi}(0,\boldsymbol{r})a(\boldsymbol{r}-\boldsymbol{r}')\widetilde{\phi}(0,\boldsymbol{r}') \tag{17}$$

contains the 'noise'-terms coming from (14) [82]. It was assumed here that $\langle\phi(0,\boldsymbol{r})\rangle = 0$ and $a(\boldsymbol{r})$ denotes the initial two-point correlator

$$a(\boldsymbol{r}) := C(0,0;\boldsymbol{r}+\boldsymbol{r}',\boldsymbol{r}') = \langle\phi(0,\boldsymbol{r}+\boldsymbol{r}')\phi(0,\boldsymbol{r}')\rangle = a(-\boldsymbol{r}) \tag{18}$$

while the last relation follows from spatial translation-invariance which we shall admit throughout.

It is instructive to consider briefly the case of a free field, where $\mathcal{V}=0$. Variation of (15) with respect to $\widetilde{\phi}$ and ϕ, respectively, then leads to the equations of motion

$$2\mathcal{M}\partial_t\phi = \Delta\phi + T\widetilde{\phi} \;\;,\;\; -2\mathcal{M}\partial_t\widetilde{\phi} = \Delta\widetilde{\phi} \tag{19}$$

The first one of those might be viewed as a Langevin equation if $\widetilde{\phi}$ is interpreted as a noise. Comparison of the two equations of motion (19) shows that if the order-parameter ϕ is characterized by the 'mass' $\mathcal{M}$ (which by physical convention is positive), then the associated response field $\widetilde{\phi}$ is characterized by the *negative* mass $-\mathcal{M}$. This characterization remains valid beyond free fields.

We now concentrate on actions $J_0[\phi,\widetilde{\phi}]$ which are Galilei-invariant. This means that if $\langle . \rangle_0$ denotes the averages calculated only with the action J_0, the Bargman superselection rules [7]

$$\left\langle \underbrace{\phi\ldots\phi}_{n}\,\underbrace{\widetilde{\phi}\ldots\widetilde{\phi}}_{m} \right\rangle_0 \sim \delta_{n,m} \tag{20}$$

hold true. It follows that both response and correlation functions can be exactly expressed in terms of averages with respect to the deterministic part alone. For example (we suppress for notational simplicity the spatial coordinates) [104]

$$R(t,s) = \left.\frac{\delta\langle\phi(t)\rangle}{\delta h(s)}\right|_{h=0} = \left\langle\phi(t)\widetilde{\phi}(s)\right\rangle = \left\langle\phi(t)\widetilde{\phi}(s)\,e^{-J_b[\widetilde{\phi}]}\right\rangle_0 = \left\langle\phi(t)\widetilde{\phi}(s)\right\rangle_0 \tag{21}$$

where the 'noise' part of the action was included in the observable and the Bargman superselection rule (20) was used. In other words, *the two-time response function does not depend explicitly on the 'noise' at all.* The correlation function is reduced similarly

$$C(t,s;\boldsymbol{r}) = T\int_{\mathbb{R}_+\times\mathbb{R}^d} \mathrm{d}u\mathrm{d}\boldsymbol{R}\,\left\langle \phi(t,\boldsymbol{r}+\boldsymbol{y})\phi(s,\boldsymbol{y})\widetilde{\phi}(u,\boldsymbol{R})^2\right\rangle_0$$
$$+\frac{1}{2}\int_{\mathbb{R}^{2d}} \mathrm{d}\boldsymbol{R}\mathrm{d}\boldsymbol{R}'\,a(\boldsymbol{R}-\boldsymbol{R}')\left\langle \phi(t,\boldsymbol{r}+\boldsymbol{y})\phi(s,\boldsymbol{y})\widetilde{\phi}(0,\boldsymbol{R})\widetilde{\phi}(0,\boldsymbol{R}')\right\rangle_0 \qquad (22)$$

Only terms which depend explicitly on the 'noise' remain – recall the vanishing of the 'noiseless' two-point function $\langle\phi(t)\phi(s)\rangle_0 = 0$ because of the Bargman superselection rule.

Therefore, the dynamical symmetries of non-equilibrium kinetics are characterized by the 'deterministic' part of Langevin equation. Such deterministic non-linear diffusion/Schrödinger equations with $\mathfrak{age}_1$ or $\mathfrak{sch}_1$ as a dynamical symmetry can be explicitly constructed [119] but we shall not go into the details here. Since all quantities of interest will reduce to some kind of response function, one may calculate them from the *requirement that they transform covariantly under the action ageing subgroup* (with Lie algebra $\mathfrak{age}_d$) obtained from the Schrödinger group when leaving out time-translations. In this survey, we shall concentrate on the two-time autoresponse function $R(t,s)$ for which the requirement of covariance reduces to the two conditions $X_0R(t,s) = X_1R(t,s) = 0$. Since time-translations are not included in the ageing group, the generators X_n can be generalized from (11) to the following form

$$X_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n r\partial_r - \frac{(n+1)n}{4}\mathcal{M}t^{n-1}r^2 - \frac{x}{2}(n+1)t^n - \xi n t^n \;\;;\;\; n\geq 0 \qquad (23)$$

where ξ is a new quantum number associated with the field ϕ on which the generators X_n act. This last term can only be present for systems out of an equilibrium state (the requirement of time-translation invariance and $[X_1, X_{-1}] = 2X_0$ lead to $\xi = 0$). Solving the two differential equations for R gives the explicit form of $R(t,s)$, see (26) below.

While this discussion was carried out explicitly for the case $z = 2$, it is tempting to try and generalize this idea to more general values of z. In this way, the notion of *local scale-transformation* has been introduced, which is based on the following main assumptions [58].

(i) In principle, the following conformal time-transformations should be included

$$t\mapsto t' = \frac{\alpha t+\beta}{\gamma t+\delta} \;\;;\;\; \alpha\delta-\beta\gamma = 1 \qquad (24)$$

For applications to ageing, however, time-translations generated by β must be left out (generalizing the restriction $\mathfrak{sch}_d \to \mathfrak{age}_d$).

(ii) The generator X_0 of scale-transformations is

$$X_0 = -t\partial_t - \frac{1}{z}r\partial_r - \frac{x}{z} \qquad (25)$$

Table 1. Magnetic systems quenched into the coexistence phase ($T < T_c$) which satisfy (26) with the exponents $a = a'$ and λ_R. d is the spatial dimension and the numbers in brackets estimate the numerical uncertainty in the last digit(s). In the spherical model, long-range initial conditions are included and in the long-range spherical model the exchange couplings decay as $J_{\boldsymbol{r}} \sim |\boldsymbol{r}|^{-d-\sigma}$. In the bond-disordered Ising model, the couplings are taken homogeneously from the interval $[1-\varepsilon/2, 1+\varepsilon/2]$. Then $z = z(T,\varepsilon) = 2+\varepsilon/T$ exactly [100, 101] and one observes roughly $1.3 \lesssim \lambda_R(T,\varepsilon) \lesssim 1.7$.

model	d	z	$a = a'$	λ_R		Ref.
Ising	2	2	1/2	1.26(1)		[61]
	2	2	$\simeq 0.5$	1.24(2)		[86, 80]
	3	2	1/2	1.60(2)		[61]
Potts-3	2	2	0.49	1.19(3)		[86, 80]
Potts-8	2	2	0.51	1.25(1)		[86, 80]
XY	3	2	0.5	1.7		[2]
XY spin wave	≥ 2	2	$d/2-1$	d	angular response	[104]
spherical	> 2	2	$d/2-1$	$(d-\alpha)/2$	$C_{\rm ini}(\boldsymbol{r}) \sim \lvert\boldsymbol{r}\rvert^{-d-\alpha}$	[94, 103]
long-range	> 2	σ	$d/\sigma-1$	$d/2$	$0<\sigma<2$	
spherical	≤ 2	σ	$d/\sigma-1$	$d/2$	$0<\sigma<d$	[27]
diluted Ising	2	$2+\varepsilon/T$	$1/z(T,\varepsilon)$	$\lambda_R(T,\varepsilon)$	disordered	[72]

where x is the scaling dimension of the quasi-primary operator on which X_0 is supposed to act. Physically, this implies that there is a single relevant length scale $L(t) \sim t^{1/z}$.

(iii) Spatial translation-invariance is required.

Generators for infinitesimal local scale-transformations have been explicitly constructed and it can be shown that for any value of z there is a linear invariant equation, analogous to (6) [58]. *Local scale-invariance* (LSI) assumes in particular that the two-time response functions transform covariantly under these local scale-transformations, hence $X_0 R = X_1 R = 0$. This leads to the prediction [58, 104, 69, 71]

$$R(t,s) = \left\langle \phi(t)\widetilde{\phi}(s)\right\rangle = s^{-1-a} f_R(t,s) \;\;,\;\; f_R(y) = f_0\, y^{1+a'-\lambda_R/z}(y-1)^{-1-a'} \quad (26)$$

where the exponents $a, a', \lambda_R/z$ are related to $x, \xi, \widetilde{x}, \widetilde{\xi}$ and f_0 is a normalization constant.‡

Starting with [57], the prediction (26) has been reproduced in many different spin systems and we list examples quenched to below criticality in table 1 and quenched

‡ We point out that the prediction (26) as well as the explicit form (23) of X_n, valid for $z = 2$, assume that the mean order-parameter $\langle\phi(0,\boldsymbol{r})\rangle = m_0 = 0$ at the initial moment when the quench to $T < T_c$ or $T = T_c$ is made.

Table 2. Systems quenched to a critical point of their stationary state which satisfy (26) with the exponents a, a' and λ_R/z. d is the spatial dimension and the numbers in brackets estimate the uncertainty in the last digit(s). CSM stands for the spherical model with a conserved order-parameter, FA denotes the Frederikson-Andersen model, NEKIM is the non-equilibrium kinetic Ising model and BCP and BPCP denote the bosonic contact and pair-contact processes (see eqs. (45,48) for the definitions of the control parameter α and of α_C), respectively. In the spherical model, long-range initial correlations $C_{\rm ini}(\boldsymbol{r}) \sim |\boldsymbol{r}|^{-d-\alpha}$ were considered. If $d+\alpha > 2$, these reduce to short-ranged initial correlations (denoted S), but for $d+\alpha < 2$ a new class L arises. In those models described by a Langevin equation, one has used throughout, with the exception of the CSM, the simple white noise $\langle \eta(t,\boldsymbol{r})\eta(s,\boldsymbol{r}')\rangle = 2T\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-s)$.

model	d	a	$a'-a$	λ_R/z		Ref.
random walk		-1	0	0		[34]
OJK-model		$(d-1)/2$	$-1/2$	$d/4$		[17, 91, 69]
Ising	1	0	$-1/2$	$1/2$		[50, 85, 62]
	2	0.115	$-0.187(20)$	0.732(5)		[108, 71]
	3	0.506	$-0.022(5)$	1.36		[108, 71]
XY	3	0.52	0	1.34(5)		[2]
spherical $d>2$	<4	$d/2-1$	0	$d/2-\alpha/4-1/2$	L	[103]
	>4	$d/2-1$	0	$(d-\alpha)/4+1/2$	L	[103]
	<4	$d/2-1$	0	$3d/4-1$	S	[51]
	>4	$d/2-1$	0	$d/2$	S	[51]
CSM	>2	$d/4-1$	0	$(d+2)/4$		[14]
disordered Ising	$4-\varepsilon$	$1-\frac{1}{2}\sqrt{\frac{6\varepsilon}{53}}$	0	$3-\frac{1}{2}\sqrt{\frac{6\varepsilon}{53}}$	O(ε), log	[23, 115, 116]
FA	>2	$1+d/2$	-2	$2+d/2$		[90]
	1	1	$-3/2$	2		[90, 88]
Ising spin glass	3	0.060(4)	$-0.76(3)$	0.38(2)		[66, 69]
contact process	1	-0.681	$+0.270(10)$	1.76(5)	$t/s \gtrsim 1.1$	[41, 75, 71]
	>4	$d/2-1$	0	$d/2+2$		[109]
NEKIM	1	-0.430(4)	0	1.9(1)		[96]
BCP	≥ 1	$d/2-1$	0	$d/2$		[8]
BPCP	>2	$d/2-1$	0	$d/2$	$\alpha \leq \alpha_C$	[8]

to the critical point in table 2. For $T < T_c$, it is found empirically that $a = a'$ in all examples considered so far. We point out that agreement with local scale-invariance eq. (26) is not only obtained for systems where the dynamical exponent is $z=2$, but that rather there exist quite a few examples where z can become considerably larger or smaller than 2. It must be remembered, however, that the above derivation of (26) for a stochastic Langevin equation has for the time being only been carried out for $z=2$§

§ See section 4 for a recent extension of the method to $z=4$.

and the justification of $X_0R = X_1R = 0$ remains an open problem for $z \neq 2$ although the result (26) seems to work remarkably well. Still, it is non-trivial that a relatively simple extension of dynamical scaling should be capable of making predictions which can be reproduced in physically quite different systems.

A few comments are still needed: (i) for the XY model in the spin-wave approximation (table 1), eq. (26) holds for the response of the angular variable $\phi = \phi(t, \boldsymbol{r})$ which is related to the XY spin through $\boldsymbol{S} = (\cos\phi, \sin\phi)$. Magnetic responses have a different scaling form [18, 1]. (ii) in the critical disordered Ising model (table 2) one finds a logarithmic scaling form $R(t, s) = (r_0 + r_1 \ln(t - s)) f_R(t/s)$ [23, 115, 116] such that the computed $f_R(y)$ is consistent with (26) to one-loop order, or up to terms of order $\mathrm{O}(\varepsilon)$. (iii) Finally, a two-loop calculation of the critical non-conserved $\mathrm{O}(n)$-model does produce in $4-\varepsilon$ dimensions an expression for $f_R(y)$ which is incompatible with (26) [22] and a similar result is anticipated in $2+\varepsilon$ dimensions [45]; although the one-loop results are still compatible [21, 22, 45]. Should one conclude from these studies that for $T = T_c$ the prediction (26) and by implication local scale-invariance can only hold approximatively ? This might well be a subtle question. Deviations between (26) and the field-theoretical studies typically arise when $t/s \approx 1$. However, in this region the field-theoretical results for $f_R(y)$ do not agree with the ones of non-perturbative numerical studies [108]. Since the perturbative expansion usually carried out in field-theoretical studies does not necessarily take care of the Galilei-invariance, it is necessary to carefully check that the truncation of the ε-series does not introduce slight inaccuracies. Only after this has been done (for example by re-summing the ε-series) and checked by comparing with non-perturbative data, meaningful quantitative statements on the scaling functions can be made. (iv) Throughout, it was implicitly assumed that the order-parameter vanishes initially. Systematic studies on what happens when this condition is relaxed are only now becoming available [5, 25, 45]. These extensions might be particularly important for chemical kinetics. (v) We did not include growth models here but shall discuss them in section 4.

If $z = 2$, it is also possible, using eq. (22), to derive explicit predictions for the two-time correlation function [104, 67]. These have been tested in some exactly solvable models [104, 71], the $2D$ Ising model [67] and the $2D$ q-states Potts model with $q = 2, 3, 8$ [86, 80]. Extensions to $z = 4$ have been studied very recently [110, 14], see section 4.

This survey is organized as follows. In section 2 we review results on the ageing behaviour of several critical models with non-equilibrium steady-states. The first two models are chosen because their steady-state phase-transitions are in the paradigmatic universality classes of directed percolation (DP) and in the parity-conserving (PC) universality classes. The numerical results on these models are supplemented by the exactly solved bosonic variants of the contact and pair-contact processes. In this way it becomes clear that most aspects of the scaling description of ageing in simple magnets does carry over to this more general situation. However, a central issue, namely the definition of an universal limit fluctuation-dissipation ratio X_∞ [51] and which has

received so much recent attention in the non-equilibrium critical dynamics of magnets, see [32, 24] for reviews, does not appear to have an obvious analogue. Remarkably, the same kind of evidence in favour of a non-trivial extension of dynamical scaling towards a larger dynamical symmetry group of *local* scale-transformation previously found in magnets also appears in the models without detailed balance. In section 3, we review in more detail how the stochastic Langevin equations underlying the bosonic contact and pair-contact processes can be shown to actually possess a local scale-invariance. These evidences should form a promising basis to look for more manifestation of local scale-invariance in systems with dynamical scaling which remain always very far from equilibrium. A different class of non-equilibrium models is studied in section 4, where kinetic growth as described by the Edwards-Wilkinson and the Mullins-Herring equations is studied. To account for those models, the formulation of LSI must be generalized [58] to values $z \neq 2$ of the dynamical exponent. We consider explicitly the case $z = 4$ and apply it to the Mullins-Herring model. We conclude in section 5.

2. Ageing with absorbing steady-states

We now describe the ageing behaviour of system without an equilibrium stationary state. We shall realize this system as models of interacting classical particles, where the stochastic dynamics is such that the detailed-balance condition no longer holds. For the simple models we shall consider here it turns out that if the stationary state is not at a critical point, only a single stable stationary state remains to which the system relaxes within a finite time and no ageing is possible. For this reason, we shall study the ageing behaviour at criticality.

We remark that one might also go to non-equilibrium stationary states by considering driven systems [113]. However, the dynamics of those is more complicated than the systems at hand because of a further strong spatial anisotropy and the description in terms of local scale-invariance would require to generalize the local scale-transformations accordingly. That is beyond the scope of this survey. Another very interesting class of ageing non-equilibrium systems are zero-range process, see [43, 53] for recent reviews. Because they do not have a spatial structure which would admit a Galilei-invariance, their dynamical scaling cannot be extended to some form of local scale invariance and for this reason they are not considered here, despite their intrinsic interest.

2.1. Contact process

The contact process is a paradigmatic system for the study of non-equilibrium phase-transitions, see e.g. [74] for a review.‖ The steady-state phase-transition of the contact

‖ Recall that bifurcations arising in many simple models of mathematical biology [93] come from the mean-field treatment of the phase-transition in the contact process.

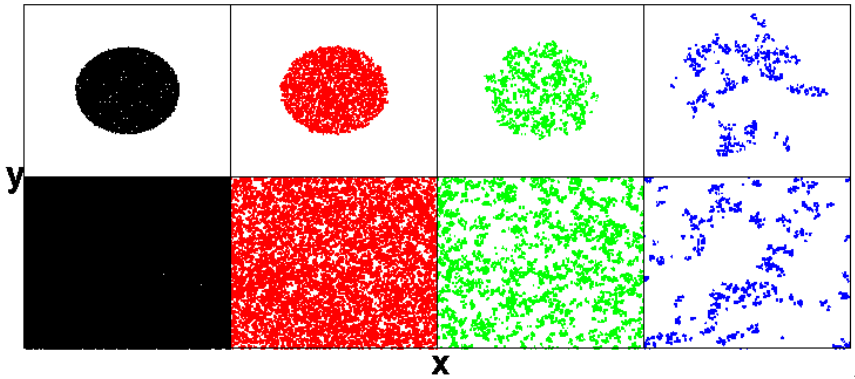

Figure 3. Microscopic evolution of clusters in the critical $2D$ contact process, on a lattice of size 1000×1000. The initial condition of the upper series is a full circle with radius 100 placed in the center of the lattice, while in the lower series it is a full lattice. The times are $t = [2, 20, 200, 2000]$ for the upper series and $t = [20, 200, 2000, 20000]$ below. After [109].

process is in the same universality class as one of the transitions of the celebrated Ziff-Gulari-Barshad model [128], which is meant to describe the catalytic reaction $2\mathrm{CO} + \mathrm{O}_2 \longrightarrow 2\mathrm{CO}_2$. The model may be defined in terms of a time-dependent discrete variable $n_i(t) \in \{0, 1\}$, defined on each site i of a hypercubic lattice, which describe configurations of particles and empty sites. The dynamics is defined as follows: for each time-step, select randomly a site i of the lattice. If i is occupied (i.e. $n_i = 1$), that particle vanishes with probability p. Otherwise, with probability $1-p$ a new particle is created on one of the nearest neighbours of i, chosen at random and provided that chosen site is still empty. Formally, this may be expressed through the reactions $A \to \emptyset$ and $A \to 2A$, with rates corresponding to p and $1-p$, respectively. In the steady-state, the model has a continuous phase-transition at some critical value p_c. Numerically, $p_c = 0.2326746(5)$ in $1D$ and $p_c = 0.37753(1)$ in $2D$.

A first characteristic of the dynamics of the critical contact process can be seen by looking at the temporal evolution of certain initial configurations, see figure 3. In contrast to magnetic systems, see figure 2 for comparison, in the contact process there is no apparent growing length-scale at all and the evolution proceeds via the slow dissolution of the particle clusters. Cluster dilution had first been demonstrated to occur in several variants of the two-dimensional voter model [38] but also occurs in the early stages of surface ageing in simple magnets [106, 107].

In studies of the ageing behaviour, one goes beyond the average particle-density $N(t) := \langle n_i(t)\rangle \sim t^{-\delta}$ at criticality $p = p_c$. Define the two-time (connected) autocorrelator and autoresponse functions

$$C(t,s) := \langle n_i(t) n_i(s)\rangle - \langle n_i(t)\rangle\langle n_i(s)\rangle \;\;,\;\; R(t,s) := \left.\frac{\delta\langle n_i(t)\rangle}{\delta h_i(s)}\right|_{h=0} \tag{27}$$

where $h_i(s)$ is the rate of the spontaneous creation process $\emptyset \to A$ at the site i at time

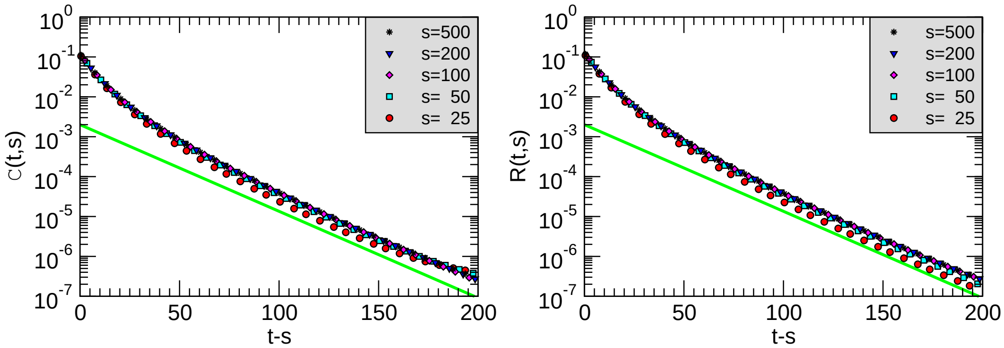

Figure 4. Connected autocorrelation function $C(t, s)$ and autoresponse function $R(t, s)$ of the $1D$ contact process in the active phase ($p = 0.1$). The straight lines are proportional to $\exp(-0.05(t - s))$. After [41].

s. These may be calculated in a standard fashion either from simulations [109, 75] or else from the transfer-matrix renormalization group [41]. Recent field-theoretical calculations [13] will also be described.

2.1.1. Active phase In contrast with simple magnets, where there are two distinct stable ground states in the low-temperature phase, in the active phase of the contact process there is only a *single* stable steady state. Consequently, there is here *no* breaking of time-translation invariance and we illustrate this in $1D$ in figure 4. After a short transient, the data for both $C(t, s)$ and $R(t, s)$ collapse when plotted over against $t - s$ which means that the contact process shows no ageing in its active phase.

2.1.2. Absorbing phase From the comparison with the high-temperature phase of simple magnets, one would also expect to find time-translation invariance in the absorbing phase of the contact process. However, the correlation function shows a subtlety the origin of which is best understood by considering the case $p = 1$ first. If $p = 1$, particles on different sites are uncorrelated and simply decay with a fixed rate. For any fixed site i and with two times $t > s$, it is clear that $n_i(t)n_i(s) = n_i(t)$, since $n_i \in \{0, 1\}$. Hence $\langle n_i(t)n_i(s)\rangle = \langle n_i(t)\rangle$ and $C(t, s) = N(t)(1 - N(s))$. For sufficiently long times, $C(t, s)$ will then only depend on t. Indeed, this behaviour survives in the entire absorbing phase [41]. On the other hand, the expected time-translation invariance for the autoresponse function is readily checked.

2.1.3. Critical point For the critical contact process we show in figure 5a that ageing does occur, that is, the autocorrelation and the autoresponse depend on *both* the observation time t and the waiting time s. Furthermore, when the same data are re-plotted over against t/s, a data collapse after rescaling can be achieved, see figure 5b. Lattices with a large initial particle density $n \approx 0.8 - 1$ were used [41, 109, 75]. This is different with respect to the magnetic systems of section 1, where the order-parameter

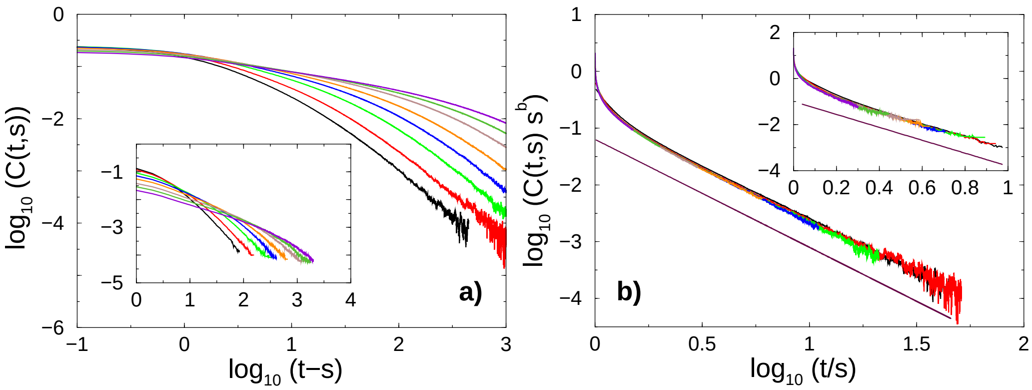

Figure 5. Connected autocorrelation function of the critical contact process in $1D$ (main plots) and $2D$ (insets). Panel (a) shows the ageing of the autocorrelation functi … ıt lines correspond to the …

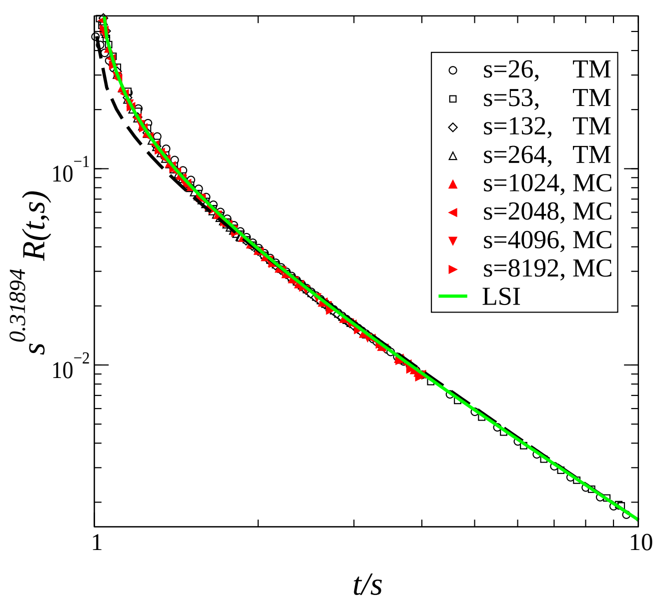

Figure 6. Autoresponse function for the critical $1D$ contact process for several waiting times s. The data labelled TM come from the transfer matrix renormalization group [41] and MC denotes Monte Carlo data.

had a vanishing initial value. By analogy with simple magnets, one defines the ageing exponents a, b and the autocorrelation and autoresponse exponents $\lambda_{C,R}$ from

$$
\begin{aligned}
C(t,s) &= s^{-b} f_C(t/s) \quad , \quad f_C(y) \sim y^{-\lambda_C/z} \\
R(t,s) &= s^{-1-a} f_R(t/s) \quad , \quad f_R(y) \sim y^{-\lambda_R/z}
\end{aligned} \tag{28}
$$

where the asymptotic forms should hold for $y \to \infty$. Similarly, scaling can be observed for the autoresponse function as shown in figure 6. On the other hand, the unconnected correlator behaves for large times simply as $\langle n_i(t) n_i(s) \rangle \sim (ts)^{-b/2}$.

The results for the ageing exponents $a, b, \lambda_C, \lambda_R$ are collected in table 3. The

Table 3. Nonequilibrium exponents for the contact process (CP), the non-equilibrium kinetic Ising model (NEKIM), the bosonic contact process (BCP) and the bosonic pair-contact process (BPCP). Several kinetic growth models based on the Edwards-Wilkinson (EW) and Mullins-Herring (MH) equations are also listed, see section 4.

model	d	a	b	λ_C/z	λ_R/z	method	Ref.
CP	1	$-0.68(5)$	$0.32(5)$	$1.85(10)$	$1.85(10)$	TMRG	[41]
		$-0.57(10)$	0.319	$1.9(1)$	$1.9(1)$	Monte Carlo	[109]
		-0.681			$1.76(5)$	Monte Carlo	[75]
	2	$0.3(1)$	$0.901(2)$	$2.8(3)$	$2.75(10)$	Monte Carlo	[109]
NEKIM	1	$-0.430(4)$	$0.570(4)$	$1.9(1)$	$1.9(1)$	Monte Carlo	[96]
BCP	≥ 1	$d/2-1$	$d/2-1$	$d/2$	$d/2$	exact	[8]
BPCP	> 2	$d/2-1$	$d/2-1$	$d/2$	$d/2$	exact, $\alpha < \alpha_C$	
	> 2 & < 4	$d/2-1$	0	$d/2$	$d/2$	exact, $\alpha = \alpha_C$	[8]
	> 4	$d/2-1$	$d/2-2$	$d/2$	$d/2$	exact, $\alpha = \alpha_C$	
EW2	≥ 1	$d/2-1$	$d/2-1-\rho$	$d/2-\rho$	$d/2$	exact	[110]
MH1	≥ 1	$d/4-1$	$d/4-1$	$d/4$	$d/4$	exact	[110]
MH2	≥ 1	$d/4-1$	$d/4-1-\rho/2$	$d/4-\rho/2$	$d/4$	exact	[110]
MHc	≥ 2	$(d-2)/4$	$(d-2)/4$	$(d+2)/4$	$(d+2)/4$	exact	[14]

agreement between the results of the TMRG and Monte Carlo simulations serves as a useful contrôle on the fiability of the results. While the equality $\lambda_C = \lambda_R$ is fully analogous to what was seen in non-equilibrium critical dynamics, the exponents a and b are no longer equal but satisfy

$$1 + a = b = 2\delta \tag{29}$$

Some comments are in order.

(i) If the critical contact process were a Markov process, these exponents might be calculated from the global persistence exponent θ_g through the scaling relation [87, 73, 3]

$$\frac{\lambda_C}{z} = \theta_g - \frac{2(1-d)-\eta}{2z}, \tag{30}$$

which would predict $\lambda_C/z = 1.98(2)$ in $1D$ and $\lambda_C/z = 3.5(5)$ in $2D$. Although this not too far form the values reported in table 3, the differences appear to be significant. If that conclusion is correct, it would point towards the existence of temporal long-range correlations and hence of an effective non-markovian dynamics of the critical contact process.

(ii) The relation $1 + a = b$ can be understood [13] as follows to be a consequence of the rapidity-reversal symmetry of Reggeon field-theory (which is generally thought

to be in the same universality class as the contact process). The field-theoretical action in the Janssen-de Dominics formulation reads *at* the critical point

$$\mathcal{J}[\phi,\widetilde{\phi}] = \int \mathrm{d}t\mathrm{d}\boldsymbol{r}\ \left[\widetilde{\phi}\left(\partial_t - D\Delta\right)\phi - u\left(\widetilde{\phi}-\phi\right)\widetilde{\phi}\phi - h\widetilde{\phi}\right] \tag{31}$$

where ϕ and h are the coarse-grained particle-densities and creation rates for particles. For $h=0$ and if the time $t\in\mathbb{R}$ is unbounded, the action is invariant under the *rapidity reversal*, see [121, 122]

$$\widetilde{\phi}(t,\boldsymbol{r}) \mapsto -\phi(-t,\boldsymbol{r}) \ \ , \ \ \phi(t,\boldsymbol{r}) \mapsto -\widetilde{\phi}(-t,\boldsymbol{r}) \tag{32}$$

In particular, it follows that the scaling dimensions $x_\phi = x_{\widetilde{\phi}} = \beta/\nu$ must be equal. This remains true even if rapidity-reversal is broken by initial conditions at time $t=0$. For a rapidity-reversal-invariant action $\mathcal{J}$, the connected correlator becomes [13]

$$\begin{aligned} C(t,s;\boldsymbol{r}-\boldsymbol{r}') &= \langle\phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r}')\rangle - \langle\phi(t,\boldsymbol{r})\rangle\langle\phi(s,\boldsymbol{r}')\rangle \\ &= \langle\widetilde{\phi}(-t,\boldsymbol{r})\widetilde{\phi}(-s,\boldsymbol{r}')\rangle - \langle\widetilde{\phi}(-t,\boldsymbol{r})\rangle\langle\widetilde{\phi}(-s,\boldsymbol{r}')\rangle \\ &= 0 \end{aligned} \tag{33}$$

where the second line comes from rapidity-reversal symmetry and the last line follows from causality. Hence $C(t,s;\boldsymbol{r})=0$ in the steady-state but for relaxations from an initial state $C(t,s;\boldsymbol{0}) = \langle\phi(t)\phi(s)\rangle_c$ and $R(t,s;\boldsymbol{0}) = \langle\phi(t)\widetilde{\phi}(s)\rangle$ are *non-*vanishing and have the same scaling dimensions, which implies (29) and also $\lambda_C = \lambda_R$.

While this explains the origin of eq. (29) for the contact process, it is not yet understood why it also holds true in the NEKIM [96] (see below) where rapidity-reversal symmetry is not known to be satisfied. On the other hand eq. (29) is not universally valid. In the critical *bosonic* contact process, one has $a=b$. The critical bosonic pair-contact process furnishes further examples with $a\neq b$ but the relation between a and b is distinct from eq. (29).

(iii) The non-equality of the exponents a and b is only possible in systems with non-equilibrium steady-states. Indeed, for equilibrium systems, one has time-translation invariance and combining this with the scaling forms would give

$$C(t,s) \sim (t-s)^{-b} \ \ , \ \ R(t,s) \sim (t-s)^{-1-a}$$

The fluctuation-dissipation theorem would then give $a=b$. Hence the equality $a=b$ is a necessary condition that a quasi-stationary state, which might be present for $t-s\ll 1$, is an equilibrium one.

(iv) Is it possible to define a non-equilibrium temperature from the steady-states of systems without detailed balance ? A recent attempt by Sastre, Dornic and Chaté [112] started from the observation that in simple magnets, the fluctuation-dissipation ratio $X(t,s)\to 1$ as $t\to\infty$ and $t-s\to 0$. From this observation, they define a *dynamical temperature* by

$$\frac{1}{T_{\rm dyn}} := \lim_{t\to\infty}\left(\lim_{t-s\to 0}\frac{R(t,s)}{\partial C(t,s)/\partial s}\right) \tag{34}$$

By explicit calculation, they confirm that in the $2D$ critical voter model (where indeed $a = b$) this limit exists, has a non-trivial value and is universal [112]. Still, their appealing idea has met with several criticisms. First, Mayer and Sollich [89] construct in the coarsening $1D$ Glauber-Ising model a defect-pair observable such that the fluctuation-dissipation ratio $X(t, s) \neq 1$ in the short time-regime (in particular they show $\lim_{s\to\infty} X(s, s) = 3/4$). Second, $T_{\rm dyn}$ can only be finite if $a = b$ and the examples listed in table 3 show that the fluctuation-dissipation ratio itself is in general no longer defined. It appears that the proposal (34) relies too heavily on specific properties of the voter model.

(v) Rather than the fluctuation-dissipation ratio $X(t, s)$ as defined in eq. (3) for systems with detailed balance, one may instead consider the ratio $\Xi(t, s)$, and its limit Ξ_∞, which are defined by [41]

$$\Xi(t,s) := \frac{R(t,s;\mathbf{0})}{C(t,s;\mathbf{0})} = \frac{f_R(t/s)}{f_C(t/s)} \;\; ; \;\; \Xi_\infty := \lim_{s\to\infty}\left(\lim_{t\to\infty}\Xi(t,s)\right) \tag{35}$$

which are well-defined because of (29). The limit Ξ_∞, being the ratio of two quantities with the same classical and scaling dimensions, is expected to be universal [13]. Baumann and Gambassi [13] argue that a value $\Xi(t, s)^{-1} \neq 0$ is a measure for the breaking of the rapidity-reversal symmetry (in the same way as $X \neq 1$ measures the distance from an equilibrium state) and show explicitly that only the zero-momentum modes contribute to a non-vanishing value of $\Xi(t, s)^{-1}$. In this respect, for systems with rapidity-reversal symmetry, Ξ appears to be the analogue of the fluctuation-dissipation ratio R. Of course, the applicability of their argument depends on the validity of the scaling relation $1 + a = b$. For the value of Ξ_∞, they find in $4 - \varepsilon$ dimensions

$$\Xi_\infty = 2\left[1 - \varepsilon\left(\frac{119}{480} - \frac{\pi^2}{120}\right)\right] + O(\varepsilon^2) \tag{36}$$

which in $1D$ is in semiquantitative agreement with the estimate $\Xi_\infty = 1.15(5)$ [41].

Before one can discuss the form of the scaling function $f_R(y)$, it is necessary to study the rôle of the initial conditions. Indeed, all existing simulations on the ageing in that model start from a lattice with a non-vanishing particle density and hence in contrast to simulations in magnetic system, where the initial order-parameter was set to zero. For the time-dependent order-parameter one expects the scaling form [126, 81, 82, 13]

$$\langle\phi(t)\rangle = \phi_0 t^\theta f\left(\phi_0 t^{\theta+\beta/(\nu z)}\right) \tag{37}$$

where $\phi_0 = \langle\phi(0)\rangle$ is the initial value of the order-parameter (which for the contact process is the particle density) and θ is the slip exponent. Hence there is characteristic time scale $\tau_* \sim \phi_0^{-1/(\theta+\beta/(\nu z))}$ where a change of scaling behaviour takes place. For any non-vanishing value of the dimensionful variable ϕ_0 the long-time scaling behaviour is effectively described by the $\tau_* \to 0$ limit. For a vanishing initial order-parameter the exponents $\lambda_{C,R}$ are independent of the equilibrium exponents [81]. For $\phi_0 \neq 0$, Baumann

and Gambassi show from an analysis of the scaling behaviour of both responses and correlators that [13]

$$\lambda_C = \lambda_R = d + z + \frac{\beta}{\nu} \tag{38}$$

The available exponent estimates in $1D$ and $2D$ from simulational studies [41, 109, 75, 71] agree quite well with this prediction. We point out that the available results for the limit ratio X_∞ [41, 13], see eq. (36), are found for a finite initial particle-density.

On the other hand, in the theory of LSI as reviewed in section 1 the implicit assumption $\phi_0 = 0$ was made. Since ϕ_0 couples to a relevant scaling variable, it is not obvious why a theory formulated in the limit $\phi_0 t^{\theta+\beta/(\nu z)} \to 0$ should be valid in the opposite limit $\phi_0 t^{\theta+\beta/(\nu z)} \to \infty$ where *all* existing studies on the ageing in the contact process have been performed. It is therefore remarkable that LSI with $\phi_0 = 0$ still captures well the behaviour of the $\phi_0 \neq 0$ data. This is shown in figure 6, where for almost all values of t/s an almost perfect agreement with the LSI prediction (26), derived for $\phi_0 = 0$, is found.¶ Hinrichsen [75] has argued that a more ambitious test may be performed by plotting $r(y) := f_R(y)[y^{\lambda_R/z-1-a'}(y-1)^{1+a'}]$ over against $t/s - 1$. If LSI with $\phi_0 = 0$ holds, one expects that $r(y) = \text{const}$. In this way, he found that for although the measured function $f_R(y)$ agrees nicely with eq. (26) if t/s is large enough, for values $t/s \lesssim 1.1$, the function $f_R(y)$ remains well-defined but changes to a different behaviour which is no longer described by eq. (26). The high quality of his data makes it clear that this change of behaviour in the scaling function cannot be explained away by invoking corrections to scaling. Further unpublished calculations [42] for extremely large values of s confirm these conclusions. In addition, the same conclusion has been reached from a detailed field-theoretic study of the two-time response in momentum space [13]. A possible explanation of the form of $f_R(y)$ in terms of LSI will require the extension of the theory to include a non-vanishing value of ϕ_0.

2.2. Non-equilibrium kinetic Ising model

We now review results, obtained by Ódor [96], on the ageing behaviour in a non-equilibrium kinetic Ising model (NEKIM) where the parity of the total particle-number is conserved. The model was introduced by Menyhárd [92] and combines spin-flips as in zero-temperature Glauber dynamics with spin-exchanges as in Kawasaki dynamics. The model is formulated either in terms of Ising spins ($\uparrow$, $\downarrow$) or else in terms of a particle-reaction model of the kinks with occupied or empty sites ($\bullet$, $\circ$). First, the Glauber-like part of the dynamics contains a diffusive motion

$$\uparrow\downarrow\downarrow \rightleftharpoons \uparrow\uparrow\downarrow \quad \text{or equivalently} \quad \bullet\circ \rightleftharpoons \circ\bullet \;\; ; \;\; \text{with rate } D$$

¶ In the limit $\phi_0 \to 0$, the results for $R(t,s)$ appear to be consistent with LSI [13].

and the pair-annihilation of nearest neighbours

$$\uparrow\downarrow\uparrow\rightarrow\uparrow\uparrow\uparrow \quad \text{or equivalently} \quad \bullet\bullet\rightarrow\circ\circ \;\; ; \;\; \text{with rate } 2\alpha$$

The Kawasaki-like part of the dynamics is described by

$$\uparrow\uparrow\downarrow\downarrow\rightleftharpoons\uparrow\downarrow\uparrow\downarrow \quad \text{or equivalently} \quad \circ\bullet\circ\rightleftharpoons\bullet\bullet\bullet \;\; ; \;\; \text{with rate } k$$

In full, this is a model describing branching and annihilating random walks with an even number of offspring. By increasing k, one finds a second-order phase-transition [92] where the kinks go from an absorbing to an active state. This phase-transition is in the so-called parity-conserving (PC) universality class [54, 29, 26] which is different from the one of the contact process.

Using the parameterization $k = 1 - 2\Gamma$, $D = \Gamma(1-\bar{\delta})/2$ and $2\alpha = \Gamma(1+\bar{\delta})$, the critical point is located at $\Gamma = 0.35$, $k = 0.3$ and $\bar{\delta} = -0.3928$ [96]. Measuring the kink-density through an efficient cluster algorithm, and starting from a fully ordered kink state with spins being alternatingly $\uparrow$ and $\downarrow$, he finds a nice power-law scaling $\langle n_i(t)\rangle \sim t^{-0.285(2)}$.

Next, measuring the unconnected kink-kink two-time correlation function, Ódor's data are fully compatible with the scaling behaviour $\langle n_i(t)n_i(t')\rangle \sim t'^{-0.57}(t/t')^{-0.285}$ for $t/t' \to \infty$ and allows to determine the ageing exponent b, see table 3. This result is also consistent with earlier results on the spin-spin autocorrelator in the same model, see [96] for details. The connected autocorrelator was also calculated, with the result $\lambda_C/z = \lambda_R/z = 1.9(1)$. This is in agreement with the scaling relation (38).

Finally, the spin-autoresponse $R(t,s)$ with respect to a magnetic field coupling to a spin was calculated [96], by adapting methods used previously by Hinrichsen [75] for the contact process. We recall that for the fully ordered kink initial state, the initial spin magnetization vanishes. In figure 7 it is shown that a clear data collapse is found when $1 + a = b$, as in the contact process.

Comparing the form of the scaling function $f_R(y)$ with the prediction of LSI, Ódor finds a perfect agreement as long as his data remain in the scaling limit. In particular, he carefully considered the limit $y \to 1$ (see inset in figure 7). Within the numerical accuracy, the data for the rescaled response scaling function $r(y)$ remain essentially constant for all values of t/s, as long the model is in the scaling regime. Clear deviations from a horizontal line only occur once finite-time corrections have broken dynamical scaling and by increasing s, the regime where (i) scaling holds and (ii) $r(y)$ is constant progressively extends to ever smaller values of $y = t/s$. This finding is different from the $1D$ contact process [75, 13] (see above). It is conceivable that the agreement with LSI in the NEKIM comes from the fact that the initial configuration used has a vanishing magnetization.

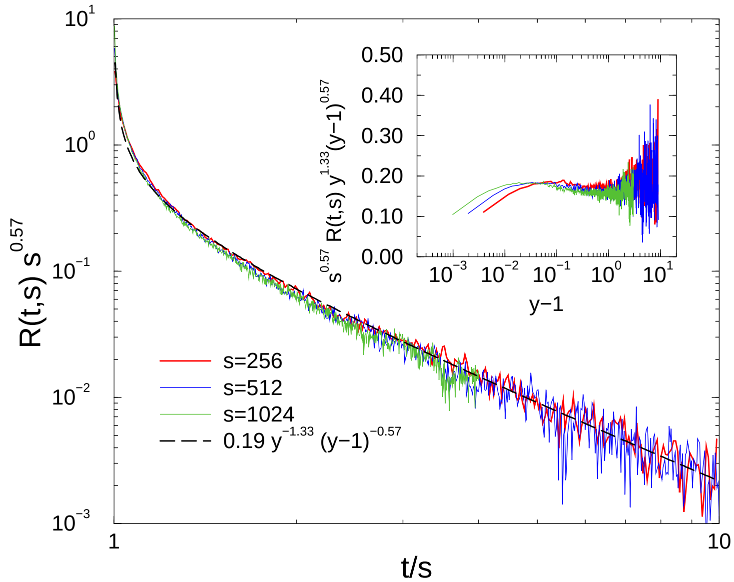

Figure 7. Scaling of the autoresponse function for the critical $1D$ NEKIM [96], for several waiting times s. The inset shows a rescaled response function, with $s = [256, 512, 1024]$ from right to left.

2.3. Bosonic contact and pair-contact processes

Two exactly solvable models allow to confirm the conclusions drawn from the above numerical studies.

The bosonic contact process was introduced in order to describe clustering phenomena in biological systems [77] whereas the bosonic pair-contact process was originally conceived [99] as a solvable variant of the usual (fermionic) pair-contact process. These models are defined as follows. Consider a set of particles of a single species A which move on the sites of a d-dimensional hypercubic lattice. On any site one may have an arbitrary (non-negative) number of particles and it is this property which makes up the difference with the usual contact and pair-contact processes considered before where on each site at most one particle is allowed. Single particles may hop to a nearest-neighbour site with unit rate and in addition, the following single-site creation and annihilation processes are admitted

$$mA \xrightarrow{\mu} (m+1)A \ \ , \ \ pA \xrightarrow{\lambda} (p-\ell)A \ \ ; \ \ \text{with rates } \mu \text{ and } \lambda \tag{39}$$

where ℓ is a positive integer such that $|\ell| \leq p$. We are interested in the following special cases:

(i) *critical bosonic contact process:* $p = m = 1$. Here only $\ell = 1$ is possible. Furthermore the creation and annihilation rates are set equal $\mu = \lambda$.

(ii) *critical bosonic pair-contact process:* $p = m = 2$. We fix $\ell = 2$, set $2\lambda = \mu$ and

define the control parameter [+]

$$\alpha := \frac{3\mu}{2D} \tag{40}$$

The dynamics of these models is conveniently described in terms creation operators $a^{\dagger}(t,\boldsymbol{r})$ of a particle at time t and location $\boldsymbol{r}$ and the corresponding annihilation operator $a(t,\boldsymbol{r})$, see [36, 117]. The equation of motion for the space-time-dependent particle-density $\rho(t,\boldsymbol{x}) := \langle a^{\dagger}(t,\boldsymbol{x})a(t,\boldsymbol{x})\rangle = \langle a(t,\boldsymbol{x})\rangle$ reads, after a rescaling $t \mapsto t/(2D)$ [77, 99]

$$\frac{\partial}{\partial t}\langle a(t,\boldsymbol{x})\rangle = \frac{1}{2}\Delta_{\boldsymbol{x}}\langle a(t,\boldsymbol{x})\rangle - \frac{\lambda\ell}{2D}\langle a(t,\boldsymbol{x})^p\rangle + \frac{\mu}{2D}\langle a(t,\boldsymbol{x})^m\rangle + h(t,\boldsymbol{x}) \tag{41}$$

where we have used the short-hand

$$\Delta_{\boldsymbol{x}} f(t,\boldsymbol{x}) := \sum_{r=1}^{d}\Big(f(t,\boldsymbol{x}-\boldsymbol{e}_r) + f(t,\boldsymbol{x}+\boldsymbol{e}_r) - 2f(t,\boldsymbol{x})\Big) \tag{42}$$

Similarly, the equal-time correlation functions satisfy the equations of motion

$$\begin{aligned}\frac{\partial}{\partial t}\langle a(\boldsymbol{x})a(\boldsymbol{y})\rangle = \frac{1}{2}\sum_{k=1}^{d}\Big[&\langle a(\boldsymbol{x})a(\boldsymbol{y}-\boldsymbol{k})\rangle + \langle a(\boldsymbol{x})a(\boldsymbol{y}+\boldsymbol{k})\rangle \\ &+ \langle a(\boldsymbol{x}-\boldsymbol{k})a(\boldsymbol{y})\rangle + \langle a(\boldsymbol{x}+\boldsymbol{k})a(\boldsymbol{y})\rangle - 4\langle a(\boldsymbol{x})a(\boldsymbol{y})\rangle\Big]\end{aligned} \tag{43}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle (a(\boldsymbol{x}))^2\right\rangle = &\sum_{k=1}^{d}\Big[\langle a(\boldsymbol{x})a(\boldsymbol{x}-\boldsymbol{k})\rangle + \langle a(\boldsymbol{x})a(\boldsymbol{x}+\boldsymbol{k})\rangle - 2\left\langle a(\boldsymbol{x})^2\right\rangle\Big] \\ &+ \frac{\mu(1+\ell)}{2D}\langle a(\boldsymbol{x})^m\rangle\end{aligned} \tag{44}$$

where in eq. (43) $\boldsymbol{x} \neq \boldsymbol{y}$ is understood. Since $\langle n(\boldsymbol{x})^2\rangle = \langle a(\boldsymbol{x})^2\rangle + \langle a(\boldsymbol{x})\rangle$, the main equal-time quantity of interest, namely the variance $\sigma^2 := \langle n(\boldsymbol{x})^2\rangle - \langle n(\boldsymbol{x})\rangle^2$ can be found.

The equations of motion (43,44) are already written for the critical line given by [99]

$$\ell\lambda = \mu. \tag{45}$$

where they naturally close. For the bosonic contact process $p = m = 1$ there is an extension to arbitrary values of λ, μ which still closes. In both models, the spatial average of the local particle-density $\rho(\boldsymbol{x},t) := \langle a(\boldsymbol{x},t)\rangle$. remains constant in time

$$\int \mathrm{d}\boldsymbol{x}\,\rho(\boldsymbol{x},t) = \int \mathrm{d}\boldsymbol{x}\,\langle a(\boldsymbol{x},t)\rangle = \rho_0 \tag{46}$$

where ρ_0 is the initial mean particle-density. Furthermore, the critical line (45) separates an active phase with a formally infinite particle-density in the steady-state from an

[+] If instead we would treat a coagulation process $2A \to A$, where $\ell = 1$, the results presented in the text are recovered by setting $\lambda = \mu$ and $\alpha = \mu/D$.

absorbing phase where the steady-state particle-density vanishes. The phase diagrams are sketched in figure 8.

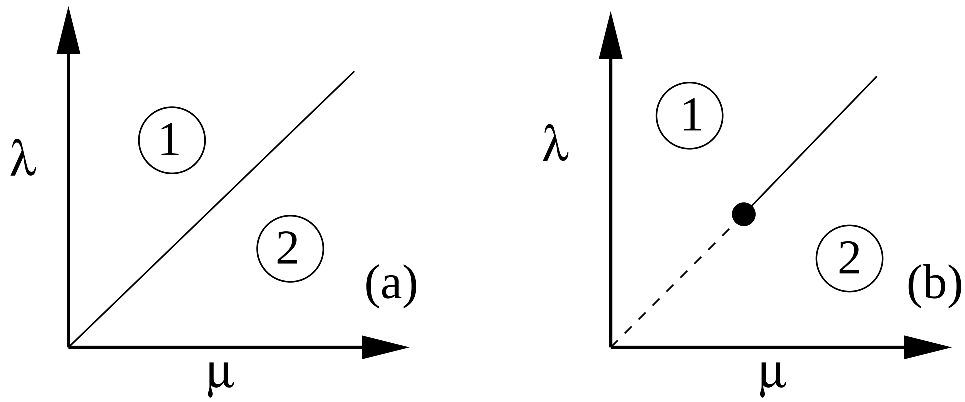

Figure 8. Schematic phase-diagrams for $D \neq 0$ of (a) the bosonic contact process and the bosonic pair-contact process in $d \leq 2$ dimensions and (b) the bosonic pair-contact process in $d > 2$ dimensions. The absorbing region 1, where $\lim_{t\to\infty} \rho(\boldsymbol{x}, t) = 0$, is separated by the critical line eq. (45) from the active region 2, where $\rho(\boldsymbol{x}, t) \to \infty$ as $t \to \infty$. Clustering along the critical line is indicated in (a) and (b) by full lines, but in the bosonic pair-contact process with $d > 2$ the steady-state may also be homogeneous (broken line in (b)). These two regimes are separated by a multicritical point.

The physical nature of this transition becomes apparent when equal-time correlations are studied [77, 99]. For example, for the bosonic contact process at criticality one has [77]

$$\left\langle a(t,\boldsymbol{x})^2 \right\rangle = \begin{cases} c_1 \; t^{-d/2+1} & ; \text{ if } d < 2 \\ c_2 \; \ln t & ; \text{ if } d = 2 \\ c_3 + c_4 \; t^{-d/2+1} & ; \text{ if } d > 2 \end{cases} \tag{47}$$

where $t \gg 1$ and $c_0, \ldots, c_4$ are known positive constants. For $d \leq 2$, the *fluctuations* in the mean particle-density increases with time, although the mean particle-density itself remains constant. Physically, this means that the particle-number on relatively few sites will increase while many other sites will become empty. Only for $d > 2$ fluctuations will eventually die out. For the bosonic contact process, this critical behaviour is the same along the entire critical line.

For the bosonic pair-contact process, that is different. Rather, there exists a critical value α_C of the control parameter, given by

$$\frac{1}{\alpha_C} = 2 \int_0^\infty \mathrm{d}u \, \left(e^{-4u} I_0(4u) \right)^d \tag{48}$$

and where $I_0(u)$ is a modified Bessel function. Specific values are $\alpha_C(3) \approx 3.99$ and

$\alpha_C(4) \approx 6.45$ and $\lim_{d\searrow 2}\alpha_C(d) = 0$. Then the variance behaves as [99]

$$\left\langle a(t,\boldsymbol{x})^2\right\rangle = \begin{cases} f_0 & ;\ \text{if } \alpha < \alpha_C \\ f_1\, t^{d/2-1} & ;\ \text{if } \alpha = \alpha_C \text{ and } 2 < d < 4 \\ f_2\, t & ;\ \text{if } \alpha = \alpha_C \text{ and } d > 4 \\ f_3\, e^{t/\tau} & ;\ \text{if } \alpha > \alpha_C \text{ (or } d < 2) \end{cases} \tag{49}$$

where the $f_0, \ldots, f_3$ and τ are known positive constants. This means that at the multicritical point at $\alpha = \alpha_C$ there occurs a clustering transition such that for $\alpha < \alpha_C$ the systems evolves towards a more or less homogeneous state while for $\alpha \geq \alpha_C$ particles accumulate on very few lattice sites while the other ones remain empty. In contrast with the bosonic contact process, clustering occurs in some region of the parameter space for all values of d.

We are interested in studying the impact of this clustering transition on the two-time correlations and linear responses. In order to obtain the equations of motion of the two-time correlator, the time-ordering of the operators $a(t,\boldsymbol{x})$ must be taken in account. This leads to the following equations of motion for the two-time correlator, after rescaling the times $t \mapsto t/(2D)$, $s \mapsto s/(2D)$, and for $t > s$, [49, 8]

$$\begin{aligned} &\frac{\partial}{\partial t}\left\langle a(t,\boldsymbol{x})a(s,\boldsymbol{y})\right\rangle \\ &= \frac{1}{2}\Delta_{\boldsymbol{x}}\left\langle a(t,\boldsymbol{x})a(s,\boldsymbol{y})\right\rangle - \frac{\lambda\ell}{2D}\left\langle a(t,\boldsymbol{x})^p a(s,\boldsymbol{y})\right\rangle + \frac{\mu}{2D}\left\langle a(t,\boldsymbol{x})^m a(s,\boldsymbol{y})\right\rangle \end{aligned} \tag{50}$$

We are interested in the two-time connected correlation function*

$$C(t,s;\boldsymbol{r}) := \left\langle a(t,\boldsymbol{x})a(s,\boldsymbol{x}+\boldsymbol{r})\right\rangle - \rho_0^2 \tag{51}$$

and take an uncorrelated initial state, hence $C(0,0;\boldsymbol{r}) = 0$. The linear two-time response function is found by adding a particle-creation term $\sum_{\boldsymbol{x}} h(\boldsymbol{x},t)\left(a^\dagger(\boldsymbol{x},t) - 1\right)$ to the quantum hamiltonian H and taking the functional derivative

$$R(t,s;\boldsymbol{r}) := \left.\frac{\delta\langle a(t,\boldsymbol{r}+\boldsymbol{x})\rangle}{\delta h(s,\boldsymbol{x})}\right|_{h=0} \tag{52}$$

and for which the usual scaling behaviour (28) is anticipated.

The solution of the equations (50) for the two-time quantities is now straightforward, if just a little tedious. It can be shown [8] that the anticipated scaling behaviour (28) exists along the critical line, but for the BPCP the further condition $\alpha \leq \alpha_C$ is required. For these cases, the exponents are listed in table 3. In particular, we see that at the multicritical point $\alpha = \alpha_C$, the exponents a and b are different and furthermore, do *not* satisfy the relation $1+a = b$ found for the critical contact process and the NEKIM. This means that there is no straightforward way to define an analogue of a

* It can be shown that in the scaling regime $C(t,s;\boldsymbol{r}) \simeq \langle n(t,\boldsymbol{r}_0)n(s,\boldsymbol{r}+\boldsymbol{r}_0)\rangle$ describes the two-time density-density correlation function [12].

			$f_R(y)$	$f_C(y)$
contact process BCP			$(y-1)^{-\frac{d}{2}}$	$(y-1)^{-\frac{d}{2}+1}-(y+1)^{-\frac{d}{2}+1}$
pair contact process	$\alpha<\alpha_C$	$d>2$	$(y-1)^{-\frac{d}{2}}$	$(y-1)^{-\frac{d}{2}+1}-(y+1)^{-\frac{d}{2}+1}$
	$\alpha=\alpha_C$	$2<d<4$	$(y-1)^{-\frac{d}{2}}$	$(y+1)^{-\frac{d}{2}}{}_2F_1\left(\frac{d}{2},\frac{d}{2};\frac{d}{2}+1;\frac{2}{y+1}\right)$
		$d>4$	$(y-1)^{-\frac{d}{2}}$	$(y+1)^{-\frac{d}{2}+2}-(y-1)^{-\frac{d}{2}+2}+(d-4)(y-1)^{-\frac{d}{2}+1}$
Edwards-Wilkinson EW2			$(y-1)^{-\frac{d}{2}}$	$(y-1)^{-\frac{d}{2}+1+\rho}-(y+1)^{-\frac{d}{2}+1+\rho}$
Mullins-Herring MH1			$(y-1)^{-\frac{d}{4}}$	$(y-1)^{-\frac{d}{4}+1}-(y+1)^{-\frac{d}{4}+1}$
Mullins-Herring MH2			$(y-1)^{-\frac{d}{4}}$	$(y-1)^{-\frac{d}{4}+1+\rho/2}-(y+1)^{-\frac{d}{4}+1+\rho/2}$
Mullins-Herring MHc			$(y-1)^{-(d+2)/4}$	$(y-1)^{-(d-2)/4}-(y+1)^{-(d-2)/4}$

Table 4. Scaling functions (up to normalization) of the autoresponse and autocorrelation of the critical bosonic contact and bosonic pair-contact processes [8]. The scaling functions for some simple growth models [110, 14], as defined in section 4, are also listed.

limit fluctuation-dissipation ratio for particle-reaction models without detailed balance. Furthermore, the explicit form of the scaling functions can also be found and are listed in table 4. While the form of the autoresponse function $f_R(y)=(y-1)^{-d/2}$ is remarkably simple, the results for the autocorrelation function can be rendered as an integral

$$f_C(y)=\mathcal{C}_0\int_0^1 \mathrm{d}\theta\, \theta^{a-b}(y+1-2\theta)^{-d/2} \tag{53}$$

where the exponents a,b are taken from table 3.

In section 3, we shall show how these results for $f_R(y)$ and $f_C(y)$ in the BCP and the BPCP can be understood using local scale-invariance. In section 4, we shall define the EW and MH models and, after having briefly reviewed LSI for $z=4$, then perform a similar analysis.

3. The bosonic processes and local scale-invariance

We now show that the exact results for response and correlation functions of the BCP and the BPCP as listed in table 4 can be understood from local scale-invariance [9, 11].

3.1. Bosonic contact process

The master equation which describes the critical bosonic contact process can be turned into a field-theory in a standard fashion through an operator formalism which uses a particle annihilation operator $a(t,\boldsymbol{r})$ and its conjugate $a^\dagger(t,\boldsymbol{r})$ [36, 121]. For calculating connected correlators, it is useful to define the shifted fields

$$\phi(t,\boldsymbol{r}) := a(t,\boldsymbol{r})-\rho_0$$

field	scaling dimension	mass
ϕ	x	$\mathcal{M}$
$\widetilde{\phi}$	$\widetilde{x}$	$-\mathcal{M}$
$\widetilde{\phi}^2$	$\widetilde{x}_2$	$-2\mathcal{M}$
$\Upsilon := \widetilde{\phi}^2\phi$	x_Υ	$-\mathcal{M}$
$\Sigma := \widetilde{\phi}^3\phi$	x_Σ	$-2\mathcal{M}$
$\Gamma := \widetilde{\phi}^3\phi^2$	x_Γ	$-\mathcal{M}$

Table 5. Scaling dimensions and masses of some composite fields.

$$\widetilde{\phi}(t,\boldsymbol{r}) := \bar{a}(t,\boldsymbol{r}) = a^\dagger(t,\boldsymbol{r}) - 1 \tag{54}$$

such that $\langle\phi(t,\boldsymbol{r})\rangle = 0$ (our notation implies a mapping between operators and quantum fields, using the known equivalence between the operator formalism and the path-integral formulation [39, 78, 121]). As we shall see, these fields ϕ and $\widetilde{\phi}$ will become the natural quasi-primary fields from the point of view of local scale-invariance. We remark that the response function is not affected by this shift, since

$$R(t,s;\boldsymbol{r},\boldsymbol{r}') = \left.\frac{\delta\langle a(t,\boldsymbol{r})\rangle}{\delta h(s,\boldsymbol{r}')}\right|_{h=0} = \left.\frac{\delta\langle \phi(t,\boldsymbol{r})\rangle}{\delta h(s,\boldsymbol{r}')}\right|_{h=0} \tag{55}$$

As for magnets, the field-theoretical action [78] is again decomposed $J[\phi,\widetilde{\phi}] = J_0[\phi,\widetilde{\phi}] + J_b[\phi,\widetilde{\phi}]$ into a 'deterministic' part

$$J_0[\phi,\widetilde{\phi}] := \int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u \, \left[\widetilde{\phi}(2\mathcal{M}\partial_u - \nabla^2)\phi\right] \tag{56}$$

and which is manifestly Galilei-invariant, whereas the 'noise' is described by

$$J_b[\phi,\widetilde{\phi}] := -\mu \int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u \, \left[\widetilde{\phi}^2(\phi+\rho_0)\right]. \tag{57}$$

We use uncorrelated initial conditions $C(0,0;\boldsymbol{r}) = 0$ throughout.

In what follows, some composite fields will be needed, which we list, together with their scaling dimensions and their masses, in table 5. We remark that for free fields one has

$$\widetilde{x}_2 = 2\widetilde{x} \;\;,\;\; x_\Upsilon = 2\widetilde{x} + x \;\;,\;\; x_\Sigma = 3\widetilde{x} + x \;\;,\;\; x_\Gamma = 3\widetilde{x} + 2x \tag{58}$$

but these relations need no longer hold for interacting fields. On the other hand, from the Bargman superselection rules we expect that the masses of the composite fields as given in table 5 should remain valid for interacting fields as well.

As in section 1, we now have a similar reduction to averages of the noiseless theory. First, for the computation of the response function, we add the term

$\int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u\, \widetilde{\phi}(u, \boldsymbol{R}) h(u, \boldsymbol{R})$ to the action. As usual the response function is [9]

$$
\begin{aligned}
R(t, s; \boldsymbol{r}, \boldsymbol{r}') &= \left\langle \phi(t, \boldsymbol{r}) \widetilde{\phi}(s, \boldsymbol{r}') \right\rangle \\
&= \left\langle \phi(t, \boldsymbol{r}) \widetilde{\phi}(s, \boldsymbol{r}') \exp\left(-\mu \int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u\, \widetilde{\phi}^2(u, \boldsymbol{R}) (\phi(u, \boldsymbol{R}) + \rho_0) \right) \right\rangle_0 \\
&= \left\langle \phi(t, \boldsymbol{r}) \widetilde{\phi}(s, \boldsymbol{r}') \right\rangle_0 = R_0(t, s; \boldsymbol{r}, \boldsymbol{r}')
\end{aligned} \tag{59}
$$

where we expanded the exponential and applied the Bargman superselection rule. Indeed, the two-time response is just given by the response of the (gaussian) noiseless theory. We have therefore reproduced the exact result of table 4 for the response function of the critical bosonic contact process.

Second, we have for the correlator

$$
\begin{aligned}
C(t, s; \boldsymbol{r}, \boldsymbol{r}') = \Bigg\langle \phi(t, \boldsymbol{r}) \phi(s, \boldsymbol{r}') &\exp\left(-\mu \int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u\, \widetilde{\phi}^2(u, \boldsymbol{R}) \phi(u, \boldsymbol{R}) \right) \\
&\times \exp\left(-\mu\rho_0 \int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u\, \widetilde{\phi}^2(\boldsymbol{R}, u) \right) \Bigg\rangle_0
\end{aligned} \tag{60}
$$

Expanding both exponentials separately and using the Bargman superselection rule (20) it follows that $C = C_1 + C_2$ can be written as the sums of two terms which read

$$
C_1(t, s; \boldsymbol{r}, \boldsymbol{r}') = -\mu\rho_0 \int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u\, \left\langle \phi(t, \boldsymbol{r}) \phi(s, \boldsymbol{r}') \widetilde{\phi}^2(u, \boldsymbol{R}) \right\rangle_0 \tag{61}
$$

and

$$
C_2(t, s; \boldsymbol{r}, \boldsymbol{r}') = \frac{\mu^2}{2} \int \mathrm{d}\boldsymbol{R} \mathrm{d}\boldsymbol{R}' \int \mathrm{d}u \mathrm{d}u'\, \langle \phi(t, \boldsymbol{r}) \phi(s, \boldsymbol{r}') \Upsilon(u, \boldsymbol{R}) \Upsilon(u', \boldsymbol{R}') \rangle_0 \tag{62}
$$

using the field Υ, see table 5. Hence the connected correlator is determined by three- and four-point functions of the noiseless theory.

The noiseless three-point response needed for C_1 can be found from its covariance under the ageing algebra [9]

$$
\begin{aligned}
\langle \phi(t, \boldsymbol{r}) \phi(s, \boldsymbol{r}') \widetilde{\phi}^2(u, \boldsymbol{R}) \rangle_0 &= (t-s)^{x - \frac{1}{2}\tilde{x}_2} (t-u)^{-\frac{1}{2}\tilde{x}_2} (s-u)^{-\frac{1}{2}\tilde{x}_2} \\
\times \exp\left(-\frac{\mathcal{M}}{2} \frac{(\boldsymbol{r} - \boldsymbol{R})^2}{t-u} - \frac{\mathcal{M}}{2} \frac{(\boldsymbol{r}' - \boldsymbol{R})^2}{s-u} \right) &\Psi_3(u_1, v_1) \Theta(t-u) \Theta(s-u)
\end{aligned} \tag{63}
$$

with

$$
\begin{aligned}
u_1 &= \frac{u}{t} \cdot \frac{[(s-u)(\boldsymbol{r} - \boldsymbol{R}) - (t-u)(\boldsymbol{r}' - \boldsymbol{R})]^2}{(t-u)(s-u)^2} \\
v_1 &= \frac{u}{s} \cdot \frac{[(s-u)(\boldsymbol{r} - \boldsymbol{R}) - (t-u)(\boldsymbol{r}' - \boldsymbol{R})]^2}{(t-u)^2(s-u)}
\end{aligned} \tag{64}
$$

and an undetermined scaling function Ψ_3. The Θ-functions express causality. Specifically, for the autocorrelator, i.e. $\boldsymbol{r} = \boldsymbol{r}'$ this yields, with $y = t/s$

$$\begin{aligned} C_1(t,s) &= -\mu\rho_0\, s^{-x-\frac{1}{2}\tilde{x}_2+\frac{d}{2}+1}\cdot(y-1)^{-(x-\frac{1}{2}\tilde{x}_2)} \\ &\times \int_0^1 \mathrm{d}\theta\,(y-\theta)^{-\frac{1}{2}\tilde{x}_2}(1-\theta)^{-\frac{1}{2}\tilde{x}_2}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{R}\,\exp\left(-\frac{\mathcal{M}}{2}\boldsymbol{R}^2\frac{y+1-2\theta}{(y-\theta)(1-\theta)}\right) \\ &\times H\left(\frac{\theta}{y}\frac{\boldsymbol{R}^2(y-1)^2}{(y-\theta)(1-\theta)^2},\theta\,\frac{\boldsymbol{R}^2(y-1)^2}{(y-\theta)^2(1-\theta)}\right) \end{aligned} \tag{65}$$

where H is an undetermined scaling function. A similar, but quite lengthy, expression can be derived for C_2 and depends on $\tilde{x}_2$ and x_Υ [9]. Since the critical BCP is described by a free field-theory, one can expect from table 3 that $x = \widetilde{x} = d/2$ and hence for the composite fields

$$\tilde{x}_2 = d, \qquad x_\Upsilon = \frac{3}{2}d \tag{66}$$

Consequently, the autocorrelator takes the general form

$$C(t,s) = s^{1-d/2}g_1(t/s) + s^{2-d}g_2(t/s) \tag{67}$$

For d larger than the lower critical dimension $d_* = 2$, the second term merely furnishes a finite-time correction. On the other hand, for $d < d_* = 2$, it would be the dominant one and we can only achieve agreement by discarding the scaling function g_2. It remains to be seen that g_1 is compatible with the exact result given in table 4.

This can be achieved by choosing in eq. (63) [104]

$$\Psi_3(u_1,v_1) = \Xi\left(\frac{1}{u_1}-\frac{1}{v_1}\right) \tag{68}$$

where Ξ remains an arbitrary function. Then

$$\begin{aligned} C_1(t,s) =& -\mu\rho_0 s^{\frac{d}{2}+1-x-\frac{1}{2}\tilde{x}_2}(t/s-1)^{\frac{1}{2}\tilde{x}_2-x-\frac{d}{2}} \\ &\times\int_0^1 \mathrm{d}\theta\,[(t/s-\theta)(1-\theta)]^{\frac{d}{2}-\frac{1}{2}\tilde{x}_2}\phi_1\left(\frac{t/s+1-2\theta}{t/s-1}\right) \end{aligned} \tag{69}$$

where the function ϕ_1 is defined by

$$\phi_1(w) = \int \mathrm{d}\boldsymbol{R}\,\exp\left(-\frac{\mathcal{M}w}{2}\boldsymbol{R}^2\right)\Xi(\boldsymbol{R}^2) \tag{70}$$

Now the result for the BCP in table 4 is recovered if one chooses [104, 9] $\phi_1(w) = \phi_{0,c}w^{-1-a}$. This form for $\phi_1(w)$ guarantees that the three-point response function $\langle\phi(\boldsymbol{r},t)\phi(\boldsymbol{r},s)\phi^2(\boldsymbol{r}',u)\rangle_0$ is nonsingular for $t = s$.♯

♯ We remark that for $2 < d < 4$, the same form of the autocorrelation function is also found in the critical voter-model [37].

3.2. Bosonic pair-contact process

The construction of the action follows standard lines [78]. Making the same shift eq. (54) as before, the action becomes $J[\phi,\widetilde{\phi}] = J_0[\phi,\widetilde{\phi}] + J_b[\phi,\widetilde{\phi}]$ where the 'deterministic' part now reads

$$J_0[\phi,\widetilde{\phi}] := \int \mathrm{d}\boldsymbol{r} \int \mathrm{d}t \, \left[\widetilde{\phi}(2\mathcal{M}\partial_t - \nabla^2)\phi - \alpha\widetilde{\phi}^2\phi^2\right]. \tag{71}$$

The remaining part is the noise-term which reads

$$J_b[\phi,\widetilde{\phi}] = \int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u \, \left[-\alpha\rho_0^2\widetilde{\phi}^2 - 2\alpha\rho_0\widetilde{\phi}^2\phi - \mu\widetilde{\phi}^3\phi^2 - 2\mu\rho_0\widetilde{\phi}^3\phi - \rho_0^2\widetilde{\phi}^3\right] \tag{72}$$

The discussion of the Schrödinger- or, more precisely, the ageing-invariance of J_0 can no longer use the representations we considered so far, since the equation of motion associated to J_0 is non-linear, viz.

$$2\mathcal{M}\partial_t\phi(\boldsymbol{x},t) = \nabla^2\phi(\boldsymbol{x},t) - g\phi^2(\boldsymbol{x},t)\widetilde{\phi}(\boldsymbol{x},t) \tag{73}$$

While for a constant g the well-known symmetries of this equation are those encountered before, it was pointed out recently that g rather should be considered as a dimensionful quantity and hence should transform under local scale-transformations as well [119]. The requires an extension of the generators of $\mathfrak{age}_d$ which do contain a dimensionful coupling g [9]. Then it can be shown that *the Bargman superselection rules (20) still apply* and the response function of the noiseless theory now reads [9]

$$R_0(t,s;\boldsymbol{r},\boldsymbol{r}') = (t-s)^{-\frac{1}{2}(x_1+x_2)} \left(\frac{t}{s}\right)^{-\frac{1}{2}(x_1-x_2)} \times \exp\left(-\frac{\mathcal{M}}{2}\frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{t-s}\right) \tilde{\Psi}_2\left(\frac{t}{s}\cdot\frac{t-s}{g^{1/y}}, \frac{g}{(t-s)^y}\right) \tag{74}$$

with an undetermined scaling function $\tilde{\Psi}_2$. In these calculation, we have assumed for technical simplicity that each field φ_i has a coupling g_i and only at the end, we let $g_1 = \ldots = g_n = g$. This form is clearly consistent with the results for the BPCP listed in table 4, for both $\alpha < \alpha_c$ and $\alpha = \alpha_c$, if we identify

$$x := x_1 = x_2 = a + 1 = \frac{d}{2} \;\; , \;\; \tilde{\Psi}_2 = \text{const.} \tag{75}$$

In distinction with the bosonic contact process, the symmetries of the noiseless part S_0 do not fix the response function completely but leave a certain degree of flexibility in form of the scaling function $\tilde{\Psi}_2$.

As before, averages can be reduced to averages within the 'deterministic' theory only. Technically, calculations become a little longer for the BPCP, since because of the structure of J_b several composite field must be defined. We refer to [9] for the details and merely quote here the results. First, the response function does not depend explicitly on the noise, viz.

$$R(t,s;\boldsymbol{r},\boldsymbol{r}') = R_0(t,s;\boldsymbol{r},\boldsymbol{r}') \tag{76}$$

Second, the results of table 4 for $f_C(y)$ can be reproduced from the single term

$$C_1(t,s) = \alpha\rho_0^2 \int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u \; \left\langle \phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r})\widetilde{\phi}^2(u,\boldsymbol{R})\right\rangle_0 \tag{77}$$

The required three-point function now reads

$$\begin{aligned}\left\langle \phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r}')\tilde{\phi}^2(u,\boldsymbol{R})\right\rangle_0 &= (t-s)^{x-\frac{1}{2}\tilde{x}_2}(t-u)^{-\frac{1}{2}\tilde{x}_2}(s-u)^{-\frac{1}{2}\tilde{x}_2} \\ &\times \exp\left(-\frac{\mathcal{M}}{2}\frac{(\boldsymbol{r}-\boldsymbol{R})^2}{t-u} - \frac{\mathcal{M}}{2}\frac{(\boldsymbol{r}'-\boldsymbol{R})^2}{s-u}\right)\tilde{\Psi}_3(u_1,v_1,\beta_1,\beta_2,\beta_3)\end{aligned} \tag{78}$$

with

$$u_1 = \frac{u}{t}\cdot\frac{[(s-u)(\boldsymbol{r}-\boldsymbol{R})-(t-u)(\boldsymbol{r}'-\boldsymbol{R})]^2}{(t-u)(s-u)^2} \tag{79}$$

$$v_1 = \frac{u}{s}\cdot\frac{[(s-u)(\boldsymbol{r}-\boldsymbol{R})-(t-u)(\boldsymbol{r}'-\boldsymbol{R})]^2}{(t-u)^2(s-u)} \tag{80}$$

$$\beta_1 = \frac{1}{s_2}\cdot\frac{\alpha^{1/y}}{(t-u)^2},\quad \beta_2 = \frac{1}{s_2}\cdot\frac{\alpha^{1/y}}{(s-u)^2},\quad \beta_3 = \alpha^{1/y}s_2 \tag{81}$$

$$s_2 = \frac{1}{t-u}+\frac{1}{u} \tag{82}$$

Next, we choose the following realization for $\tilde{\Psi}_3$

$$\tilde{\Psi}_3(u_1,v_1,\beta_1,\beta_2,\beta_3) = \Xi\left(\frac{1}{u_1}-\frac{1}{v_1}\right)\left[-\frac{(\sqrt{\beta_1}-\sqrt{\beta_2})\sqrt{\beta_3}}{\beta_3-\sqrt{\beta_2\beta_3}}\right]^{(a-b)} \tag{83}$$

where the scaling function Ξ was already encountered in eq. (68) for the bosonic contact process. We now have to distinguish the two different cases $\alpha < \alpha_C$ and $\alpha = \alpha_C$. For the first case $\alpha < \alpha_C$, we have $a-b=0$ so that the last factor in (83) disappears and we simply return to the expressions already found for the bosonic contact process, in agreement with the known exact results. However, at the multicritical point $\alpha = \alpha_C$ we have $a-b \neq 0$ and the last factor becomes important. We point out that only the presence or absence of this factor distinguishes the cases $\alpha < \alpha_C$ and $\alpha = \alpha_C$.

Inserting the values of the $\beta_{1,2,3}$ we finally obtain for the autocorrelation function

$$\begin{aligned}C_1(t,s) &= s^{-b}(y-1)^{(b-a)-a-1}\int_0^1 \mathrm{d}\theta\,[(t/s-\theta)(1-\theta)]^{a-b} \\ &\times \phi_1\left(\frac{t/s+1-2\theta}{t/s-1}\right)\left[\frac{\theta(t/s-1)}{(t/s-\theta)(1-\theta)}\right]^{a-b}\end{aligned} \tag{84}$$

where we have identified

$$\tilde{x}_2 = 2(b-a)+d \tag{85}$$

$C_1(t,s)$ reduces to the expression (53) if we choose the same $\phi_1(w) = \phi_{0,c}w^{-1-a}$ as before. Hence all scaling functions for the BPCP are reproduced correctly.

4. Growth models

As a further illustration on a different class of system we now describe recent work by Röthlein, Baumann and Pleimling [110] on kinetic growth models. As a simple model for ballistic deposition consider the Family model [44]. A particle is randomly dropped onto the sites of a lattice. However, before it is fixed, the particle explores the sites around the one it arrived at (typically the nearest neighbours) and fixes itself at the lattice site with the lowest height. One obtains in this way a growing surface which may be described in terms of a height variable $h(t, \boldsymbol{r})$. Clearly, since there is only irreversible deposition, the system will never arrive at an equilibrium state.

Some of the simplest continuum descriptions of these phenomena can be cast into stochastic linear equations for the height variable. For the sake of simplicity, we shall always work in the frame co-moving with the mean surface height. For example, if the deposition is purely diffusive and without mass conservation, the simplest model is the well-known Edwards-Wilkinson (EW) model [40]

$$\partial_t h(t,\boldsymbol{r}) = D\nabla^2 h(t,\boldsymbol{r}) + \eta(t,\boldsymbol{r}) \tag{86}$$

but on the other side, if mass conservation must be taken into account, one might rather consider the Mullins-Herring (MH) model, see [127]

$$\partial_t h(t,\boldsymbol{r}) = -D(\nabla^2)^2 h(t,\boldsymbol{r}) + \eta(t,\boldsymbol{r}) \tag{87}$$

Following [110], the following types of gaussian noise with vanishing first moment $\langle\eta(t,\boldsymbol{r})\rangle = 0$ will be considered:

(a) non-conserved, short-ranged $\langle\eta(t,\boldsymbol{r})\eta(s,\boldsymbol{r}')\rangle = 2D\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-s)$.

(b) non-conserved, long-range $\langle\eta(t,\boldsymbol{r})\eta(s,\boldsymbol{r}')\rangle = 2D|\boldsymbol{r}-\boldsymbol{r}'|^{2\rho-d}\delta(t-s)$ and $0<\rho<d/2$.

(c) conserved, short-ranged $\langle\eta(t,\boldsymbol{r})\eta(s,\boldsymbol{r}')\rangle = -2D\nabla_{\boldsymbol{r}}^2\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-s)$.

Then the following models were studied in [110]:

(i) EW1: eq. (86) with the non-conserved noise (a).

(ii) EW2: eq. (86) with the non-conserved, long-ranged noise (b).

(iii) MH1: eq. (87) with the non-conserved noise (a).

(iv) MH2: eq. (87) with the non-conserved, long-ranged noise (b).

(v) MHc: eq. (87) with the conserved noise (c) [14].

In the models EW1 and MHc the noise is in agreement with detailed balance while for the other models it is not. Solving the linear equations (86) and (87) is straightforward. The two-time correlation and response functions are seen to obey the same kind of scaling behaviour as for the non-equilibrium models considered before. In table 3 the values of the exponents are listed and the scaling functions for the autoresponse and

the autocorrelation are included in table 4 (the models BCP and EW1 lead to identical results and are not listed separately). The result quoted for the MHc model is only valid for $d > 2$ as stated; for $d = 2$ the scaling function becomes $f_C(y) = 2D\ln[(y-1)/(y+1)]$ [14]. Detailed simulations show that the correlation and response functions of the Family model [44] and of a variant of it are perfectly described by the EW1 model and hence should be in the same universality class [110].

We shall see shortly that all results about the scaling functions as listed in table 4 can be understood from an extension of dynamical scaling to local scale-invariance, for both the EW models, where the dynamical exponent $z = 2$, as well as for the MH models, where $z = 4$. In this context, it is instructive to consider the space-time responses $R(t, s; \boldsymbol{r})$ as well. In general, one finds the structure

$$R(t, s; \boldsymbol{r}) = R(t, s)\Phi(|\boldsymbol{r}|(t-s)^{-1/z}) \tag{88}$$

where $R(t, s) = R(t, s; \boldsymbol{0})$ is the autoresponse function. If $z = 2$, one has a simple exponential form $\Phi(u) = \exp(-\mathcal{M}u)$ [56]†† while for $z = 4$, the form of $\Phi(u)$ is more complicated. For example, in the MHc model with conserved noise [14]

$$\Phi(u) = \Phi_0 \left[{}_0F_2\left(\frac{1}{2}, \frac{d}{4}; \frac{u^4}{256}\right) - \frac{8}{d}\frac{\Gamma\left(\frac{d}{4}+1\right)}{\Gamma\left(\frac{d}{4}+\frac{1}{2}\right)}\sqrt{\frac{u^4}{256}}\; {}_0F_2\left(\frac{3}{2}, \frac{d}{4}+\frac{1}{2}; \frac{u^4}{256}\right)\right] \tag{89}$$

where Φ_0 is a known constant and ${}_0F_2$ is a hypergeometric function. Very similar expressions have been derived in the other MH models, see [110] for details.

Understanding the form of the spatio-temporal response allows for an explicit test of one of the important ingredients of the theory of local scale-invariance, namely Galilei-invariance for $z = 2$ and its generalization if $z \neq 2$. Since in the EW models one has $z = 2$, the calculation of $R(t, s; \boldsymbol{r})$ is a direct extension of the discussion presented in section 1. Since this has been discussed in detail in the litterature [56, 58] we shall not repeat it here and rather concentrate on the case $z = 4$, following [58, 110, 14].

Consider the dynamical symmetries of the 'Schrödinger operator'

$$\mathcal{S}_4 := -\lambda\partial_t + \frac{1}{16}\left(\nabla_{\boldsymbol{r}}^2\right)^2. \tag{90}$$

which will become related to the deterministic part of the Mullins-Herring equation. As before, we ask if dynamical scaling can be extended to a larger set of local scale-transformations, given standard dynamical scaling and spatial translation-invariance. Specializing the construction of [58] to $z = 4$, these generators read as follows, with the shorthands $\boldsymbol{r}\cdot\partial_{\boldsymbol{r}} := \sum_{k=1}^d r_k\,\partial_{r_k}, \nabla_{\boldsymbol{r}}^2 := \sum_{k=1}^d \partial_{r_k}^2$ and $\boldsymbol{r}^2 := \sum_{k=1}^d r_k^2$,

††For detailed quantitative tests in Ising and Potts models, see [61, 86].

$$
\begin{aligned}
X_{-1} &:= -\partial_t \\
X_0 &:= -t\partial_t - \frac{1}{4}\,\boldsymbol{r}\cdot\partial_{\boldsymbol{r}} - \frac{x}{4} \\
X_1 &:= -t^2\partial_t - \frac{x}{2}t - \lambda\boldsymbol{r}^2\,(\nabla_{\boldsymbol{r}}^2)^{-1} - \frac{1}{2}\,t\boldsymbol{r}\cdot\partial_{\boldsymbol{r}} \\
&\quad + 4\gamma\,(\boldsymbol{r}\cdot\partial_{\boldsymbol{r}})\left(\nabla_{\boldsymbol{r}}^2\right)^{-2} + 2\gamma(d-4)\left(\nabla_{\boldsymbol{r}}^2\right)^{-2} \\
R^{(i,j)} &:= r_i\partial_{r_j} - r_j\partial_{r_i}\ ; \qquad\qquad \text{where } 1\leq i<j\leq d \\
Y_{-1/4}^{(i)} &= -\partial_{r_i} \\
Y_{3/4}^{(i)} &= -t\partial_{r_i} - 4\lambda r_i\left(\nabla_{\boldsymbol{r}}^2\right)^{-1} + 8\gamma\,\partial_{r_i}\left(\nabla_{\boldsymbol{r}}^2\right)^{-2}
\end{aligned}
\tag{91}
$$

where x is the scaling dimension of the fields on which these generators act and γ, λ are further field-dependent parameters. Here, the generators $X_{\pm 1,0}$ correspond to projective changes in the time $t \mapsto (\alpha t+\beta)/(\gamma t+\delta)$ with $\alpha\delta-\beta\gamma=1$, the generators $Y_{n-1/4}^{(i)}$ are space-translations, generalized Galilei-transformations and so on and $R^{(i,j)}$ are spatial rotations. Here we use the following properties $\partial_{\boldsymbol{r}}^{\alpha}\partial_{\boldsymbol{r}}^{\beta} = \partial_{\boldsymbol{r}}^{\alpha+\beta}$ and $[\partial_{\boldsymbol{r}}^{\alpha}, r] = \alpha\partial_{\boldsymbol{r}}^{\alpha-1}$, which can be proven for fractional derivatives with extra distributional terms [58, Appendix A] which in turn are closely related to fractional derivatives as defined in [48]. Furthermore, $(\nabla_{\boldsymbol{r}}^2)^{-2} = (\nabla_{\boldsymbol{r}}^2)^{-1}\cdot(\nabla_{\boldsymbol{r}}^2)^{-1}$ and the operator $(\nabla_{\boldsymbol{r}}^2)^{-1}$ is defined, e.g., for $d=2$, by formal expansion [110, 14]

$$
(\nabla_{\boldsymbol{r}}^2)^{-1} := (\partial_{r_1}^2 + \partial_{r_2}^2)^{-1} := \sum_{n=0}^{\infty}(-1)^n\partial_{r_1}^{-2-2n}\partial_{r_2}^{2n} \tag{92}
$$

This implies the commutator $[(\nabla_{\boldsymbol{r}}^2)^n, r_i] = n\,\partial_{r_i}(\nabla_{\boldsymbol{r}}^2)^{n-1}$ for all $n\in\mathbb{Z}$.

That the generators eq. (91) indeed describe dynamical symmetries of the 'Schrödinger operator' (90) now follows from the commutators [58, 14]

$$
[\mathcal{S}_4, Y_{-1/4}^{(i)}] \;=\; [\mathcal{S}_4, Y_{3/4}^{(i)}] \;=\; [\mathcal{S}_4, R^{(i,j)}] \;=\; 0 \;\;,\;\; [\mathcal{S}_4, X_0] = -\mathcal{S}_4 \tag{93}
$$

This means that for a solution of the 'Schrödinger equation' $\mathcal{S}_4\phi=0$ the transformed function $\mathcal{X}\phi$ is again solution of the 'Schrödinger equation'. Finally

$$
[\mathcal{S}_4, X_1] = -2t\mathcal{S}_4 + \frac{\lambda}{2}\left(x - \frac{d}{2} - 1 + \frac{2\gamma}{\lambda}\right) \tag{94}
$$

hence a dynamical symmetry is obtained if the field ϕ has the scaling dimension

$$
x = \frac{d}{2} + 1 - \frac{2\gamma}{\lambda} \tag{95}
$$

Generalizing from conformal or Schrödinger-invariance, quasiprimary fields transform covariantly under the generators (91) and in particular the response function will satisfy the conditions $X_1R = X_0R = Y_{-1/4}^{(i)}R = R^{(i,j)}R = 0$ (the other conditions then follow

from the Jacobi identities) and is now characterized by its scaling dimension x_i and the further parameters γ_i, λ_i. In calculating the response function $R = \langle\phi\widetilde{\phi}\rangle$ this leads to the conditions $\lambda = -\widetilde{\lambda}$ and $\gamma = -\widetilde{\gamma}$ whereas the scaling function $\Phi(u)$ from eq. (88) can be found by solving the differential equation

$$\left(\partial_u\left(\frac{1}{u^{d-1}}\partial_u\left(u^{d-1}\partial_u\right)\right)^2 + 4\lambda u\left(\frac{1}{u^{d-1}}\partial_u\left(u^{d-1}\partial_u\right)\right) - 16\gamma\partial_u\right)\overline{\Phi}(u) = 0 \qquad (96)$$

where

$$\Phi(u) = \left(\frac{1}{u^{d-1}}\partial_u\left(u^{d-1}\partial_u\right)\right)^2\overline{\Phi}(u) \qquad (97)$$

Solving this via series expansion techniques [58, 110, 14] and checking carefully that all independent solutions are taken into account, one can indeed recover the explicit result (89) for the MHc model as a special case [14] and, similarly, also for the MH1 and MH2 models [110].

Lastly, since the MH equation is linear, there is a natural Wick theorem which allows to go over from the stochastic Langevin equation to the deterministic equation in quite an analogous way as previously for $z = 2$ [110, 14]. The extension of the technique to non-linear cases and/or to $z \neq 2, 4$ is work in progress and will be reported elsewhere [15]. In a similar way one may also check that the correlation functions agree with LSI.

The MH models considered in this section are, together with the critical spherical model with a conserved order-parameter [14], the first analytically solved examples with $z \neq 2$ where local scale-invariance could be fully confirmed. These examples make it in particular clear that the height of the surface in growth processes is a natural candidate for being described by a quasiprimary scaling operator of local scale-invariance.

5. Conclusions

In this survey, we have reviewed to what extent one may expect that a phenomenological description, which has been successfully applied to describe the ageing of magnetic systems relaxing towards equilibrium steady-states, may be extended to more general models where the stationary states are no longer part of an equilibrium statistical ensemble. This situation frequently arises in chemical kinetics, see [4, 6, 31, 124, 125], and the consideration of such systems is of interest in studies of ageing in chemical/biological systems where already the intrinsic microscopic dynamics and/or constraints does not admit relaxation towards thermal equilibrium. It is clear that the study of ageing phenomena without detailed balance still stands at its very beginning and many open questions remain. In particular, the few models reviewed here certainly do not exhaust all possibilities for ageing behaviour without detailed balance but should be rather seen as case studies whose results might suggest further research problems. This review has already served a useful purpose if it encourages people to explore more

systematically the properties of two-time observables of non-equilibrium systems, e.g. reaction-diffusion problems or biologically motivated models.

Specifically, the following points should be noted:

(i) the generic scaling form (28) was seen to be satisfied in all models considered. However, the exponent relation $a = b$, known to hold for critical systems with detailed balance, is no longer valid in general, see table 3. For different universality classes, the relation $a = b$ is either maintained or else broken in different ways. This means that there is no obvious and general analogue of an universal limit fluctuation-dissipation ratio X_∞ (for magnets one sometimes tries to relate this to a non-equilibrium temperature, see e.g. [33]) even if such an analogy may be defined for certain subclasses.

(ii) for the uncorrelated initial states which have been considered so far, one observes that the autocorrelation and autoresponse exponents agree $\lambda_C = \lambda_R$. It would be interesting to see if spatial or temporal disorder in the rates may change that conclusion, as it apparently happens in diluted magnets [114, 72].

(iii) one of our main question with respect to ageing systems has been if dynamical scaling permits an extension to a larger group of local scale-transformation [58]. It has turned out that from the point of view of LSI the responses are the most easy quantities to study. The long list of examples, see tables 1 and 2, where the two-time autoresponse function $R(t,s)$ was concluded to be in agreement with LSI is clear evidence that LSI is indeed a successful phenomenological scheme, and this for values of the dynamical exponent z which are often far from $z = 2$ characterizing simple diffusive motion. But this ansatz remains to be proven, especially for $z \neq 2$, e.g. from some underlying stochastic Langevin equation. The exactly solvable examples we have treated suggest that the idea of splitting the Langevin equation in a 'deterministic' part with possible non-trivial dynamical symmetries and a 'noise' part which breaks those can be taken over from magnets to more general reaction-diffusion type system, although the noise terms can be considerably more general. The examples studied here also suggest that the basic physical variables of these models, such as the particle-density or the height of the surface, should be directly relatable to the quasiprimary scaling operators of LSI.

It appears to us that it should be promising to investigate more systematically the foundations and consequences of a hitherto unsuspected non-trivial dynamical symmetry in scale-invariant non-equilibrium dynamics.

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